Date: 28 May 2004, Friday
Instructor: Ali Sinan Sertöz

Math 101 Calculus – Final Exam – Solutions

Q-1) Let $f(x) = x^x$, $x > 0$.
i) Find $\lim_{x\to 0+} f(x)$ and $\lim_{x\to\infty} f(x)$.
ii) Determine the intervals where $f(x)$ increases/decreases.
iii) Determine the concavity of the graph of $y = f(x)$.
iv) Find the points where $f(x)$ takes minimum and maximum values, if any.
v) Plot the graph of $y = f(x)$.

Solution:
i) Let $\lim_{x\to 0+} x^x = A$. Then,

$$\ln A = \lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0+} x = 0.$$

Therefore $\lim_{x\to 0+} x^x = 1$. On the other hand clearly $\lim_{x\to\infty} x^x = \infty$.

ii) $f'(x) = (x^x)' = (e^{x\ln x})' = (e^{x\ln x})(\ln x + 1) = (x^x)(\ln x + 1) = 0$ when $x = 1/e$. Since $\ln x$ is an increasing function, $f'(x) < 0$ for $x < 1/e$, and $f'(x) > 0$ for $x > 1/e$.

iii) $f''(x) = (x^x(\ln x + 1))' = x^x(\ln x + 1)^2 + x^x(1/x) > 0$ for all $x > 0$, so the graph is always concave up.

iv) f has a global minimum at $x = 1/e$. No global max exists.

Q-2) City A is 8 km away from a railroad which is in the form of a straight line passing through city B. City B is 9 km away from the point D which is the nearest point on the railroad to city A. As the transportation minister you want to build a highway from city A to a point C on the railroad. The cost of transportation by the railroad is 3 million TL per km. Cost of transportation along the new highway will be 5 million TL per km. You want to choose the point C so that the total cost of transportation from city A to city B, along the route AC+CB, will be minimum. Decide where the point C should be.

Solution:
Let $f(x)$ denote the total cost of transportation when point C is x km away from point D.

$f(x) = 5\sqrt{64 + x^2} + 3(9 - x)$, $0 \le x \le 9$. $f'(x) = \frac{5x}{\sqrt{64+x^2}} - 3 = 0$ when $5x = 3\sqrt{64 + x^2}$, or equivalently when $x = 6$ for x in the domain.

Checking the values of $f(x)$ at the critical and at the end points:

$f(0) = 67$
$f(9) = 5\sqrt{145} > 5\sqrt{144} = 60$
$f(6) = 59$.

So the minimum occurs when C is 6 km from point D.

Q-3) Evaluate the following two integrals:

i) $\displaystyle\int x^2 \arctan x \, dx.$ ii) $\displaystyle\int x^3\sqrt{1+x^2}\, dx.$

Solution:

i) First letting $u = \arctan x$ and $dv = x^2 dx$ and applying by-parts we get $\displaystyle\int x^2 \arctan x \, dx = \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}\, dx.$

Now $\displaystyle\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2} \Rightarrow \int \frac{x^3}{1+x^2}\, dx = \int x\, dx - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2) + C.$

Putting these together we find $\displaystyle\int x^2 \arctan x\, dx = \frac{1}{3}x^3 \arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C.$

ii) First put $x = \tan\theta$ to obtain $\displaystyle I = \int x^3\sqrt{1+x^2}\, dx = \int \frac{\sin^3\theta}{\cos^6\theta}\, d\theta = \int \frac{(1-\cos^2\theta)\sin\theta}{\cos^6\theta}\, d\theta$
Now substitute $u = \cos\theta$ to get $\displaystyle I = \int (u^{-4} - u^{-6})\, du = -\frac{1}{3}u^{-3} + \frac{1}{5}u^{-5} + C = -\frac{1}{3}\sec^3\theta + \frac{1}{5}\sec^5\theta + C.$

The substitution $x = \tan\theta$ means that θ is in a right triangle with the leg opposite to θ is x, and the leg adjacent to θ is 1. Then the hypothenuse is $\sqrt{1+x^2}$ and $\sec\theta = \sqrt{1+x^2}$. This gives $\displaystyle I = -\frac{1}{3}(1+x^2)^{3/2} + \frac{1}{5}(1+x^2)^{5/2} + C.$

Another way of doing this is as follows: Let $u = x^2$, $dv = x\sqrt{1+x^2}$ and apply by-parts to obtain $\displaystyle\int x^3\sqrt{1+x^2}\, dx = \frac{x^2}{3}(1+x^2)^{3/2} - \frac{2}{3}\int x(1+x^2)^{3/2}\, dx$. For the second integral use the substitution $u = 1+x^2$ to get $\displaystyle\int x(1+x^2)^{3/2}\, dx = \frac{1}{2}\int u^{3/2}\, du = \frac{1}{5}u^{5/2} + C = \frac{1}{5}(1+x^2)^{5/2} + C.$
Putting these together we get $\displaystyle\int x^3\sqrt{1+x^2}\, dx = \frac{x^2}{3}(1+x^2)^{3/2} - \frac{2}{15}(1+x^2)^{5/2} + C.$

Q-4) Evaluate the integral $\displaystyle\int \frac{4x^3 - x^2 + 2x - 1}{x(x-1)(x^2+1)}\, dx.$

Solution:

$$\begin{aligned}
\frac{4x^3 - x^2 + 2x - 1}{x(x-1)(x^2+1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \\
&= \frac{1}{x} + \frac{2}{x-1} + \frac{x+1}{x^2+1} \\
&= \frac{1}{x} + \frac{2}{x-1} + \left(\frac{1}{2}\frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)
\end{aligned}$$

This then immediately gives

$$\int \frac{4x^3 - x^2 + 2x - 1}{x(x-1)(x^2+1)}\, dx = \ln|x| + 2\ln|x-1| + \frac{1}{2}\ln(x^2+1) + \arctan x + C.$$

Q-5) We have two differentiable functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$. The table below lists the values of f, f', g, g' at various points. Consider the limit

$$\lim_{x \to 0} \frac{f\,(g(x))}{g\,(f(x))}.$$

Using the table below can you calculate this limit? If *yes*, find the limit. If *not*, explain what else you need to find the limit.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	4	2	23	67
1	5	3	29	71
2	0	4	31	73
3	1	5	37	79
4	2	0	41	83
5	3	1	43	89

Solution:

First note that $f(g(0)) = f(2) = 0$ and $g(f(0)) = g(4) = 0$, so the limit is in an indeterminate form. We can apply L'Hopital's rule to calculate this limit as follows:

$$\begin{aligned}
\lim_{x \to 0} \frac{f\,(g(x))}{g\,(f(x))} &= \lim_{x \to 0} \frac{f'\,(g(x))\,g'(x)}{g'\,(f(x))\,f'(x)} \\
&= \frac{f'\,(g(0))\,g'(0)}{g'\,(f(0))\,f'(0)} \\
&= \frac{f'\,(2) \cdot 67}{g'\,(4) \cdot 23} \\
&= \frac{31 \cdot 67}{83 \cdot 23} = \frac{2077}{1909}.
\end{aligned}$$
