Math 101 Calculus – Midterm Exam I- Solutions

Q-1) Find the first derivatives of the following functions. Do not simplify your answers.

i) $f(x) = \cos(x^3 + 7x^2 + 1)^2$.

ii) $f(x) = \frac{\cos x^3}{\sin x^2}$.

iii) $f(x) = \sin \frac{1}{x} + \cos \frac{1}{x}$.

iv) $f(x) = \tan (\sec (\cos x))$.

i) $f'(x) = 2 \left[ \cos(x^3 + 7x^2 + 1) \right] \left[ -\sin(x^3 + 7x^2 + 1) \right] \left[ 3x^2 + 14 \right]$.

ii) $f'(x) = \left( -\sin^3 x \cdot 3x^2 \cdot x^2 - \cos^3 x \cdot 2x \right) / (\sin x^2)^2$.

iii) $f'(x) = -\frac{1}{x^2} \cos \frac{1}{x} - \frac{1}{x^2} \left( -\sin \frac{1}{x} \right)$.

iv) $f'(x) = \sec^2(\sec(\cos x)) \cdot \sec(x) \cdot \tan(x) \cdot (-\sin x)$.

Q-2) i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 0$ for the parameterized curve $x(t) = 9t^2 + 4t + 1$, $y(t) = t^2 + 7t + 8$.

ii) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1, 1)$ for the curve $x^3 - xy + y^3 = 1$.

i) $\frac{dx}{dt} = 18t + 4$, $\frac{dy}{dt} = 2t + 7$, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + 7}{18t + 4}$, $\frac{dy}{dx} = 7$, $\frac{dy}{dx} |_{t=0} = \frac{7}{4}$.

$\frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{-118}{(18t + 4)^2}$, $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{-118}{(18t + 4)^3}$, $\frac{d^2y}{dx^2} |_{t=0} = \frac{-59}{32}$.

ii) Implicitly differentiating the given equation gives $3x^2 - y - xy' + 3y^2y' = 0$. Putting in $x = 1$, $y = 1$ and solving for $y'$ gives $y' = -1$. Differentiating implicitly once more gives $6x - 2y' - xy'' + 6y(y')^2 + 3y^2y'' = 0$. Putting in the values and solving for $y''$ gives $y'' = -7$.

Q-3) i) Evaluate the limit $\lim_{x \to 1} \frac{x^3 - x - 1}{x^3 - 3x + 2}$.

ii) Evaluate the limit $\lim_{x \to 2} \frac{x^3 - 3x - 2}{x^3 - 4x^2 + 5x - 2}$.

iii) Find the equation of the line through the point $(3, -2)$ perpendicular to the line $9x + 8y = 17$.

iv) Find the equation of the line through the points $(1, 2)$ and $(3, 5)$.
i) \[ \lim_{x \to 1} \frac{x^3 - x^2 - x + 1}{x^3 - 3x + 2} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{(x - 1)(x + 1)} = \lim_{x \to 1} \frac{x + 1}{x + 2} = \frac{2}{3}. \]

ii) \[ \lim_{x \to 2} \frac{x^3 - 3x - 2}{x^3 - 4x^2 + 5x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 1)}{(x - 2)(x^2 - 2x + 1)} = \lim_{x \to 2} \frac{x^2 + 2x + 1}{x^2 - 2x + 1} = 9. \]

iii) \[ y = \frac{8}{9}(x - 3) - 2, \text{ or } 8x - 9y = 42. \]

iv) \[ y = \frac{5 - 2}{3 - 1}(x - 1) + 2, \text{ or } 3x - 2y = -1. \]

Q-4) Consider the function

\[ f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x^2 + x} & \text{if } x \neq 0, \\
  a & \text{if } x = 0.
\end{cases} \]

For which value of \( a \) does the derivative of \( f \) exist at \( x = 0 \)?

For that value of \( a \) calculate \( f'(0) \).

\[
 f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h^2 \sin \frac{1}{h^2 + h} - a = \lim_{h \to 0} h \sin \frac{1}{h^2 + h} - \lim_{h \to 0} \frac{a}{h} = 0 - \lim_{h \to 0} \frac{a}{h} \text{ since sinus function is bounded.}
\]

\[ f'(0) = 0 \text{ when and only when } a = 0. \]

Q-5) Water is poured at the constant rate of 11 \( m^3/min \) into a container which is in the shape of an inverted right circular cone of base radius 18 \( m \) and height 9 \( m \). How fast is the level of water rising when the water is 3 \( m \) high in the container?

Let \( r(t) \) and \( h(t) \) denote the radius and height of water in the tank at time \( t \), respectively. Then \( r(t)/h(t) = 18/9 = 2 \), so \( r(t) = 2h(t) \). Volume of water in the tank at time \( t \) is

\[ V(t) = \frac{\pi}{3} r(t)^2 h(t) = \frac{4\pi}{3} h^3(t). \]

Differentiating both sides with respect to \( t \) gives 

\[ V'(t) = 4\pi h^2(t) h'(t). \]

Finally putting in \( V' = 11 \text{ m}^3/min \) and \( h(t) = 3 \text{ m} \) and solving for \( h'(t) \) we find

\[ h'(t) = \frac{11}{36\pi} \text{ m/min}. \]