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Math 101 Calculus – Midterm Exam I- Solutions

Q-1) Find the first derivatives of the following functions. Do not simplify your answers.

i) $f(x) = \left[\cos(x^3 + 7x^2 + 1)\right]^2$. **ii)** $f(x) = \dfrac{\cos x^3}{\sin x^2}$.

iii) $f(x) = \sin \dfrac{1}{x} + \cos \dfrac{1}{x}$. **iv)** $f(x) = \tan\left(\sec(\cos x)\right)$.

i) $f'(x) = 2\left[\cos(x^3 + 7x^2 + 1)\right]\left[-\sin(x^3 + 7x^2 + 1)\right]\left[3x^2 + 14x\right]$.

ii) $f'(x) = \left(-\sin x^3 \cdot 3x^2 \cdot \sin x^2 - \cos x^3 \cdot \cos x^2 \cdot 2x\right) / \left(\sin x^2\right)^2$.

iii) $f'(x) = -\dfrac{1}{x^2}\cos\dfrac{1}{x} - \dfrac{1}{x^2}\left(-\sin\dfrac{1}{x}\right)$.

iv) $f'(x) = \sec^2\left(\sec(\cos x)\right) \cdot \sec(\cos x)\tan(\cos x) \cdot (-\sin x)$.

Q-2) i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $t = 0$ for the parameterized curve $x(t) = 9t^2 + 4t + 1$, $y(t) = t^2 + 7t + 8$.

ii) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(1, 1)$ for the curve $x^3 - xy + y^3 = 1$.

i) $\dfrac{dx}{dt} = 18t + 4$, $\dfrac{dy}{dt} = 2t + 7$, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t+7}{18t+4}$, $\dfrac{dy}{dx}\Big|_{t=0} = \dfrac{7}{4}$.

$\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{-118}{(18t+4)^2}$, $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}(dy/dx)}{dx/dt} = \dfrac{-118}{(18t+4)^3}$, $\dfrac{d^2y}{dx^2}\Big|_{t=0} = -\dfrac{59}{32}$.

ii) Implicitly differentiating the given equation gives $3x^2 - y - xy' + 3y^2y' = 0$. Putting in $x = 1$, $y = 1$ and solving for y' gives $y' = -1$. Differentiating implicitly once more gives $6x - 2y' - xy'' + 6y(y')^2 + 3y^2y'' = 0$. Putting in the values and solving for y'' gives $y'' = -7$.

Q-3) i) Evaluate the limit $\displaystyle\lim_{x\to 1} \frac{x^3 - x^2 - x + 1}{x^3 - 3x + 2}$.

ii) Evaluate the limit $\displaystyle\lim_{x\to 2} \frac{x^3 - 3x - 2}{x^3 - 4x^2 + 5x - 2}$.

iii) Find the equation of the line through the point $(3, -2)$ perpendicular to the line $9x + 8y = 17$.

iv) Find the equation of the line through the points $(1, 2)$ and $(3, 5)$.

i) $\lim_{x\to 1} \dfrac{x^3 - x^2 - x + 1}{x^3 - 3x + 2} = \lim_{x\to 1} \dfrac{(x-1)^2(x+1)}{(x-1)^2(x+2)} = \lim_{x\to 1} \dfrac{(x+1)}{(x+2)} = \dfrac{2}{3}.$

ii) $\lim_{x\to 2} \dfrac{x^3 - 3x - 2}{x^3 - 4x^2 + 5x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x^2+2x+1)}{(x-2)(x^2-2x+1)} = \lim_{x\to 2} \dfrac{(x^2+2x+1)}{(x^2-2x+1)} = 9.$

iii) $y = \dfrac{8}{9}(x-3) - 2$, or $8x - 9y = 42$.

iv) $y = \dfrac{5-2}{3-1}(x-1) + 2$, or $3x - 2y = -1$.

Q-4) Consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2+x} & \text{if } x \neq 0, \\ a & \text{if } x = 0. \end{cases}$$

For which value of a does the derivative of f exist at $x = 0$?
For that value of a calculate $f'(0)$.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h^2+h} - a}{h} \\ &= \lim_{h\to 0} h \sin \frac{1}{h^2+h} - \lim_{h\to 0} \frac{a}{h} \\ &= 0 - \lim_{h\to 0} \frac{a}{h} \quad \text{since sinus function is bounded} \\ &= 0 \quad \text{when and only when } a = 0. \end{aligned}$$

Q-5) Water is poured at the constant rate of 11 m^3/min into a container which is in the shape of an inverted right circular cone of base radius 18 m and height 9 m. How fast is the level of water rising when the water is 3 m high in the container?

Let $r(t)$ and $h(t)$ denote the radius and height of water in the tank at time t, respectively. Then $r(t)/h(t) = 18/9 = 2$, so $r(t) = 2h(t)$. Volume of water in the tank at time t is $V(t) = (\pi/3)r^2(t)h(t) = (4\pi/3)h^3(t)$. Differentiating both sides with respect to t gives $V'(t) = 4\pi h^2(t)h'(t)$. Finally putting in $V' = 11$ m^3/min and $h(t) = 3$ m and solving for $h'(t)$ we find $h'(t) = \dfrac{11}{36\pi}$ m/min.