Math 101
Final Exam
Solutions

1) Let f and g be two differentiable functions defined on $\mathbb{R}$ such that

$$f(0) = 1, \quad f(1) = 1, \quad g(0) = 1, \quad \text{and} \quad g(1) = 0$$

and

$$f'(0) = 5, \quad f'(1) = 7, \quad g'(0) = 13, \quad \text{and} \quad g'(1) = 11.$$

Define

$$\phi(x) = (f \circ g)(x) \quad \text{and} \quad \rho(x) = \int_0^{f(x)} (g' \circ f)(u)\, du.$$

Find the following:

a) $\phi'(0) =?$

Solution: By Chain Rule, $\phi'(x) = f'(g(x))g'(x)$. Hence

$$\phi'(0) = f'(g(0))g'(0) = f'(1)g'(0) = 7 \cdot 13 = 91$$

b) $\rho'(1) =?$

Solution: Define

$$\psi(w) = \int_0^{w} (g' \circ f)(u)\, du.$$

Then we have

$$\rho(x) = (\psi \circ f)(x),$$

by chain rule we have

$$\rho'(x) = \psi'(f(x))f'(x),$$

and by Fundamental Theorem of Calculus Part 1 we have

$$\psi'(w) = (g' \circ f)(w)$$

Hence we have

$$\rho'(x) = (g' \circ f)(f(x))f'(x)$$

In particular we have

$$\rho'(1) = (g' \circ f)(f(1))f'(1) = g'(1)f'(1) = 11 \cdot 7 = 77.$$

2) Evaluate the integral

$$\int \frac{dx}{(x^2+x+1)^2}$$

Solution: Define $w = \frac{2}{\sqrt{3}}(x+\frac{1}{2})$. Then

$$\int \frac{dx}{(x^2+x+1)^2} = \int \frac{dx}{\left(x^2+x+\frac{1}{4}+\frac{3}{4}\right)^2} = \int \frac{dx}{\left((x+\frac{1}{2})^2+\frac{3}{4}\right)^2} =$$

$$= \int \frac{\frac{\sqrt{3}}{2}}{\left(\frac{3}{4}w^2+\frac{3}{4}\right)^2}\ dw = \int \frac{\frac{\sqrt{3}}{2}}{\frac{9}{16}\left(w^2+1\right)^2}\ dw = \frac{8}{3\sqrt{3}}\int \frac{dw}{(w^2+1)^2} = (*)$$

Define $\theta = \arctan(w)$. Then

$$(*) = \frac{8}{3\sqrt{3}}\int \frac{\sec^2(\theta)}{\sec^4(\theta)}\ d\theta = \frac{8}{3\sqrt{3}}\int \cos^2(\theta)\ d\theta = \frac{8}{3\sqrt{3}}\int \frac{1+\cos(2\theta)}{2}\ d\theta =$$

$$= \frac{8}{3\sqrt{3}}\left(\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right)+C = \frac{8}{3\sqrt{3}}\left(\frac{\theta}{2}+\frac{\sin(\theta)\cos(\theta)}{2}\right)+C =$$

$$= \frac{8}{3\sqrt{3}}\left(\frac{\arctan(w)}{2}+\frac{\left(\frac{w}{\sqrt{w^2+1}}\right)\left(\frac{1}{\sqrt{w^2+1}}\right)}{2}\right)+C = \frac{4}{3\sqrt{3}}\arctan(w)+\frac{4}{3\sqrt{3}}\frac{w}{w^2+1}+C =$$

$$= \frac{4}{3\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right)+\frac{2x+1}{3(x^2+x+1)}+C$$

3) Evaluate the integral

$$\int \frac{5x^3 + 7x^2 - 25x + 39}{x^4 - x^3 - 29x^2 - x - 30}\,dx$$

Solution: First notice

$$x^4 - x^3 - 29x^2 - x - 30 = (x+5)(x-6)(x^2+1)$$

Hence for some A, B, C, and D numbers we have

$$\frac{5x^3 + 7x^2 - 25x + 39}{x^4 - x^3 - 29x^2 - x - 30} = \frac{5x^3 + 7x^2 - 25x + 39}{(x+5)(x-6)(x^2+1)} = \frac{A}{x+5} + \frac{B}{x-6} + \frac{Cx+D}{x^2+1}.$$

Multiply both sides of the equation by $(x+5)(x-6)(x^2+1)$ we get

$$5x^3 + 7x^2 - 25x + 39 = A(x-6)(x^2+1) + B(x+5)(x^2+1) + (Cx+D)(x+5)(x-6)$$

Put $x = -5$, we get $-286 = -286\,A$. Hence $A = 1$.
Put $x = 6$, we get $1221 = 407\,B$. Hence $B = 3$.
Put $x = 0$, we get $39 = -6 + 15 - 30\,D$. Hence $D = -1$.
Put $x = 1$, we get $5 + 7 - 25 + 39 = -10 + 36 - 30(C-1)$. Hence $C = 1$
Hence

$$\frac{5x^3 + 7x^2 - 25x + 39}{x^4 - x^3 - 29x^2 - x - 30} = \frac{1}{x+5} + \frac{3}{x-6} + \frac{x-1}{x^2+1}.$$

Therefore

$$\int \frac{5x^3 + 7x^2 - 25x + 39}{x^4 - x^3 - 29x^2 - x - 30}\,dx = \int \left(\frac{1}{x+5} + \frac{3}{x-6} + \frac{x-1}{x^2+1}\right) = dx$$

$$= \int \left(\frac{1}{x+5} + \frac{3}{x-6} + \frac{x}{x^2+1} - \frac{1}{x^2+1}\right)\,dx =$$

$$= \ln|x+5| + 3\ln|x-6| + \frac{1}{2}\ln(x^2+1) - \arctan(x) + C$$

4) Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius R.

Solution: Assume that x denotes the radius of the inscribed right circular cylinder. Then $0 \leq x \leq R$ and the height of the right circular cylinder is $2\sqrt{R^2 - x^2}$. Hence we want to maximize the following volume function:

$$V(x) = 2\pi x^2 \sqrt{R^2 - x^2} \qquad \text{for} \quad 0 \leq x \leq R$$

Note that V is a continuous function on a closed interval $[0, R]$. Hence we can use the method for finding absolute extrema of a continuous function on a closed interval. First note that

$$V'(x) = 2\pi \left(2x\sqrt{R^2 - x^2} - \frac{x^3}{\sqrt{R^2 - x^2}} \right) = \frac{2\pi(2R^2 x - 3x^3)}{\sqrt{R^2 - x^2}}$$

On the interval $(0, R)$ the function V' is defined everywhere and $V'(x)$ is equal to 0 only when $x = \frac{R\sqrt{2}}{\sqrt{3}}$. Hence the only critical point of V is $x = \frac{R\sqrt{2}}{\sqrt{3}}$. Now $V(0) = 0$, $V(R) = 0$. Hence the maximum volume is given by the following value of V

$$V(\frac{R\sqrt{2}}{\sqrt{3}}) = \frac{4\pi R^3}{3\sqrt{3}}$$