Math 101 Spring 2009
Midterm 2 Solutions

1) Find the limit

$$\lim_{x\to 0} \frac{1}{x^4} \int_0^x \tan^3(u)\, du$$

Solution: We have

$$\lim_{x\to 0} \frac{1}{x^4} \int_0^x \tan^3(u)\, du = \lim_{x\to 0} \frac{\int_0^x \tan^3(u)\, du}{x^4} = (*)$$

We have a indeterminate form of type $\frac{0}{0}$. Let's try applying L'Hopital's Rule then by Fundamental Theorem of Calculus Part 1 we get:

$$(*) = \lim_{x\to 0} \frac{\frac{d}{dx}\left(\int_0^x \tan^3(u)\, du\right)}{\frac{d}{dx}(x^4)} = \lim_{x\to 0} \frac{\tan^3(x)}{4x^3} = \lim_{x\to 0} \frac{1}{4} \left(\frac{\sin(x)}{x}\right)^3 \frac{1}{\cos^3(x)} =$$

$$= \left(\lim_{x\to 0} \frac{1}{4}\right) \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)^3 \left(\lim_{x\to 0} \frac{1}{\cos^3(x)}\right) = \frac{1}{4} \cdot 1^3 \cdot \frac{1}{1^3} = \frac{1}{4}$$

Since the above calculation shows that the limit after (*) exists, it was possible to apply L'Hopital's Rule at that stage.

2) If they exist find the possible maximal and minimal surface area of a right circular cylinder whose volume is 54π cubic centimeters. (Note: Assume that the surface area includes the area of the circular wall and area of both of the circular ends.)

Solution: We have:
(Volume of the cylinder)$= V = \pi r^2 h = 54\pi$,
(Surface area of the cylinder) $= S = 2\pi rh + 2\pi r^2$.
From the first equation we get $h = \frac{54}{r^2}$. Hence

$$S(r) = 2\pi rh + 2\pi r^2 h = 2\pi(rh + r^2) = 2\pi \left(r\frac{54}{r^2} + r^2 \right) = 2\pi \left(\frac{54}{r} + r^2 \right)$$

for r in $(0, \infty)$. We have

$$S'(r) = 2\pi \left(\frac{-54}{r^2} + 2r \right) = 2\pi \left(\frac{2r^3 - 54}{r^2} \right)$$

Hence $S'(r)$ is 0 if an only if $r = 3$ and $S'(r)$ is continuous on the domain of S. Hence we have

Interval	S'	S
(0,3)	$-$	S is decreasing on $(0, 3]$.
3	0	$S(3) = 54\pi$.
$(3, \infty)$	$+$	S is increasing on $[3, \infty)$.

By considering the above table we can say that S has no absolute maximum value and say that the absolute minimum value of S is 54π.

3) Evaluate the integral

$$\int \left(\sin(x^2) + x^2 \sin(x^2) + \left(\ln(1+x^2)\right)^2 \right) \frac{x}{1+x^2}\, dx$$

Solution: We have

$$\int \left(\sin(x^2) + x^2 \sin(x^2) + \left(\ln(1+x^2)\right)^2 \right) \frac{x}{1+x^2}\, dx =$$

$$= \underbrace{\int \sin(x^2) x\, dx}_{A} + \underbrace{\int \left(\ln(1+x^2)\right)^2 \frac{x}{1+x^2}\, dx}_{B} = (**)$$

Assuming $u = x^2$ we have $du = 2xdx$ and $\frac{1}{2}du = xdx$

$$A = \int \sin(x^2) x\, dx = \frac{1}{2} \int \sin(u)\, du = -\frac{1}{2}\cos(u) + C = -\frac{1}{2}\cos(x^2) + C$$

Assuming $u = \ln(1+x^2)$ we have $du = \frac{2x}{1+x^2}dx$ and $\frac{1}{2}du = \frac{x}{1+x^2}dx$

$$B = \int \left(\ln(1+x^2)\right)^2 \frac{x}{1+x^2}\, dx = \frac{1}{2}\int u^2\, du = \frac{u^3}{6} + C = \frac{\left(\ln(1+x^2)\right)^3}{6} + C$$

Hence

$$(**) = A + B = -\frac{1}{2}\cos(x^2) + \frac{\left(\ln(1+x^2)\right)^3}{6} + C$$

4) The finite region bounded by the curves $y = x^2$ and $y = x^3$ in the the first quadrant is revolved about the y-axis to generate a solid. Find the volume of this solid.

Solution: We have

$$\text{(Volume of the solid)} = \int_0^1 \pi \left((\sqrt[3]{y})^2 - (\sqrt{y})^2 \right) \, dy == \pi \int_0^1 \left(y^{\frac{2}{3}} - y \right) \, dy =$$

$$= \pi \left(\frac{3}{5} y^{\frac{5}{3}} - \frac{y^2}{2} \Big|_0^1 \right) = \pi \left(\left(\frac{3}{5} - \frac{1}{2} \right) - 0 \right) = \frac{\pi}{10}$$