1. Evaluate the following limits without using L'Hôpital's Rule.

a. $\lim_{x\to 2} \frac{x^3-3x^2+4}{\sqrt{x^3+8x}-\sqrt{5x^2+4}} = \lim_{x\to 2}\left(\frac{x^3-3x^2+4}{(x^3+8x)-(5x^2+4)}\cdot\left(\sqrt{x^3+8x}+\sqrt{5x^2+4}\right)\right)$

$= \lim_{x\to 2} \frac{x^3-3x^2+4}{x^3-5x^2+8x-4} \cdot \lim_{x\to 2}\left(\sqrt{x^3+8x}+\sqrt{5x^2+4}\right)$

$= \lim_{x\to 2} \frac{(x-2)^2\cdot(x+1)}{(x-2)^2\cdot(x-1)} \cdot \left(\sqrt{24}+\sqrt{24}\right)$

$= \lim_{x\to 2} \frac{x+1}{x-1} \cdot 4\sqrt{6}$

$= 3\cdot 4\sqrt{6} = 12\sqrt{6}$

b. $\lim_{x\to 0} \frac{\cos 8x - \cos 5x\,\cos 3x}{x^2} = \lim_{x\to 0} \frac{\cos(5x+3x)-\cos 5x\cdot\cos 3x}{x^2}$

$= \lim_{x\to 0} \frac{\cos 5x\cdot\cos 3x - \sin 5x\cdot\sin 3x - \cos 5x\cdot\cos 3x}{x^2}$

$= \lim_{x\to 0}\left(-5\cdot 3\cdot \underset{\downarrow\, 1}{\frac{\sin 5x}{5x}}\cdot \underset{\downarrow\, 1}{\frac{\sin 3x}{3x}}\right) = -5\cdot 3\cdot 1\cdot 1 = -15$

2. Suppose that f is a differentiable function with $f(1) = 1/2$, $f'(1) = 3$, $f(2) = 7$, $f'(2) = -9$, $f(5) = 1/2$, $f'(5) = -4$.

a. Find an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 2$.

$$y - 7 = -9 \cdot (x-2)$$

b. Compute $f(1.1)$ approximately.

$$f(1.1) \approx f(1) + f'(1) \cdot (1.1-1) = \frac{1}{2} + 3 \cdot \frac{1}{10} = \frac{4}{5} = 0.8$$

c. Find $\left.\frac{du}{dt}\right|_{(t,u)=(2,1)}$ if u is a differentiable function of t satisfying the relation $f(tf(u)) = f(1+t^2u^2)$.

d/dt

$$f'(tf(u)) \cdot \left(1 \cdot f(u) + t f'(u) \frac{du}{dt}\right) = f'(1+t^2u^2) \cdot \left(2tu^2 + t^2 \cdot 2u \frac{du}{dt}\right)$$

$$\Big\Downarrow \; (t,u) = (2,1)$$

$$f'(2 \cdot f(1)) \cdot \left(f(1) + 2f'(1) \frac{du}{dt}\right) = f'(5) \cdot \left(4 + 8 \frac{du}{dt}\right)$$

$$\Downarrow$$

$$3 \cdot \left(\frac{1}{2} + 6 \frac{du}{dt}\right) = -4 \cdot \left(4 + 8 \frac{du}{dt}\right)$$

$$\Downarrow$$

$$\frac{du}{dt} = -\frac{7}{20} \quad \text{at} \quad (t,u) = (2,1)$$

3. A function f satisfies the following conditions:

❶ f is differentiable on $(-\infty, \infty)$.

❷ $f(x) = 5x + 4$ for $x \le -1$.

❸ $f(x) = x$ for $x \ge 1$.

a. Show that there is a real number A such that $f(A) = 0$.

$f(1) = 1 > 0 > -1 = f(-1)$

Since f is diff'ble, f is continuous on $[-1, 1]$.

Therefore, there is A in $(-1, 1)$ with $f(A) = 0$ by IVT.

b. Show that there is a real number B such that $f'(B) = 2$.

f is diff'ble, hence continuous on $(-\infty, \infty)$. In particular, f is continuous on $[-2, 2]$ and diff'ble on $(-2, 2)$.

Therefore, there is B in $(-2, 2)$ such that

$$f'(B) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{2 - (-6)}{4} = 2 \text{ by MVT.}$$

c. Give an example of a function satisfying the conditions ❶, ❷ and ❸ by explicitly defining $f(x)$ for $-1 < x < 1$. *(There are many such functions. You do not have to explain how you found your example, but you must verify that it satisfies the conditions.)*

Let $f(x) = x^3 - x^2 + 1$ for $-1 < x < 1$. Then

$$(x^3 - x^2 + 1)\Big|_{x=1} = 1 = x\Big|_{x=1} \text{ and } \frac{d}{dx}(x^3 - x^2 + 1)\Big|_{x=1} = 1 = \frac{d}{dx}x\Big|_{x=1}, \text{ and}$$

$$(x^3 - x^2 + 1)\Big|_{x=-1} = -1 = (5x + 4)\Big|_{x=-1} \text{ and } \frac{d}{dx}(x^3 - x^2 + 1)\Big|_{x=-1} = 5 = \frac{d}{dx}(5x + 4)\Big|_{x=-1}.$$

Hence f is diff'ble on $(-\infty, \infty)$.

4. At a certain moment the sides of a triangle are changing as follows:

- ❶ The length of the first side is 3 m and increasing at a rate of 5 m/s.
- ❷ The length of the second side is 8 m and increasing at a rate of 2 m/s.
- ❸ The length of the third side is 7 m and increasing at a rate of 1 m/s.

Determine how fast the angle between the first two sides is changing at this moment. Express your answer in units of °/s.

a = length of first side
b = length of second side
c = length of third side
θ = angle between the first two sides

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$\boxed{a = 3m, \ b = 8m, \ c = 7m}$

$$\Downarrow$$

$$49 = 9 + 64 - 48\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$$

$\frac{d}{dt}$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} - 2\frac{da}{dt}b\cos\theta - 2a\frac{db}{dt}\cos\theta - 2ab\cdot(-\sin\theta)\cdot\frac{d\theta}{dt}$$

$\boxed{\begin{array}{l} a = 3m, \ b = 8m, \ c = 7m \\ \frac{da}{dt} = 5\,m/s, \ \frac{db}{dt} = 2\,m/s, \ \frac{dc}{dt} = 1\,m/s \end{array}}$

$$\Downarrow$$

$$7\cdot 1 = 3\cdot 5 + 8\cdot 2 - 5\cdot 8\cdot\frac{1}{2} - 3\cdot 2\cdot\frac{1}{2} + 3\cdot 8\cdot\frac{\sqrt{3}}{2}\cdot\frac{d\theta}{dt}$$

$$\Downarrow$$

$$\frac{d\theta}{dt} = -\frac{1}{12\sqrt{3}}\ rad/s = -\frac{1}{12\sqrt{3}}\cdot\frac{180}{\pi}\ °/s = -\frac{5\sqrt{3}}{\pi}\ °/s$$

The angle between the first two sides is decreasing at a rate of $\frac{5\sqrt{3}}{\pi}$ °/s at this moment.