1. A function f that is defined and continuous for $x \neq 0$ satisfies the following conditions:

① $f(2-\sqrt{2}) = \sqrt[3]{1-1/\sqrt{2}}$, $f(2/3) = 0$, $f(2) = \sqrt[3]{2}$, $f(2+\sqrt{2}) = \sqrt[3]{1+1/\sqrt{2}}$

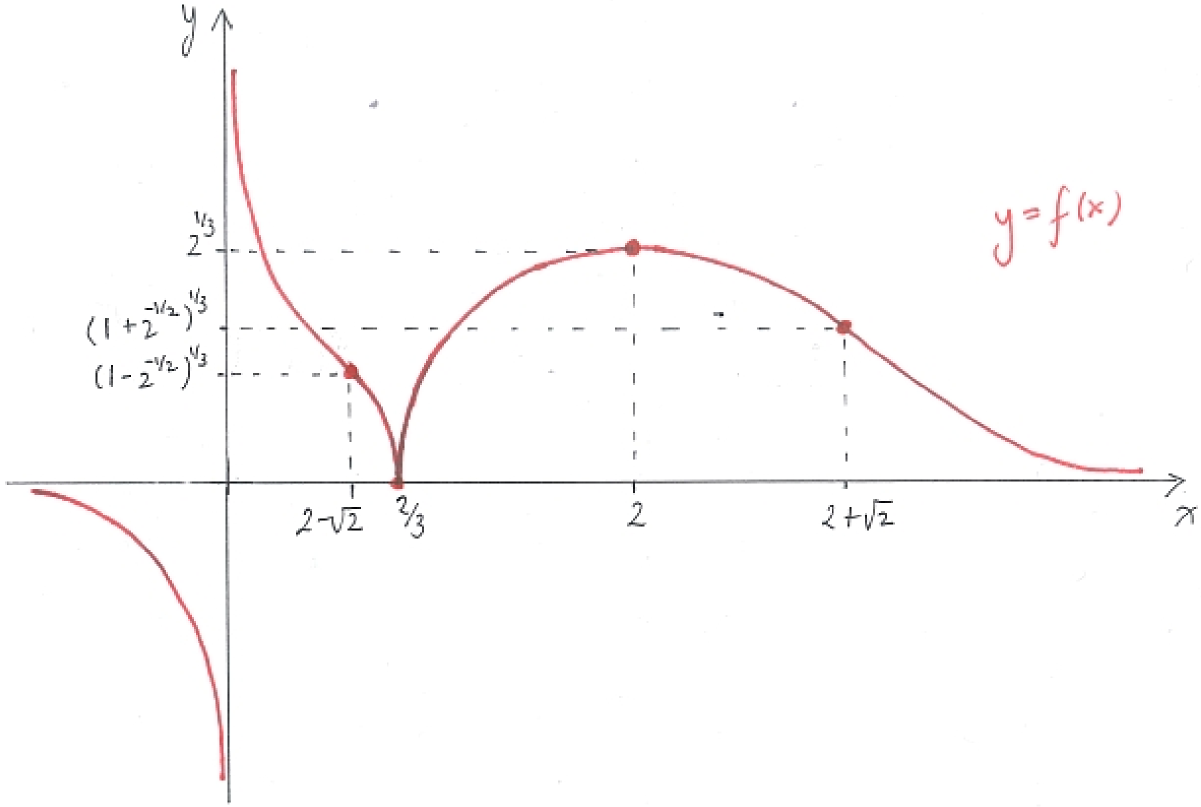
② $\lim_{x\to 0^-} f(x) = -\infty$, $\lim_{x\to 0^+} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$

③ $f'(x) < 0$ for $x < 2/3$ and $x \neq 0$, and for $x > 2$; $f'(x) > 0$ for $2/3 < x < 2$

④ $\lim_{x\to(2/3)^-} f'(x) = -\infty$, $\lim_{x\to(2/3)^+} f'(x) = \infty$

⑤ $f''(x) < 0$ for $x < 0$, and for $2-\sqrt{2} < x < 2+\sqrt{2}$ and $x \neq 2/3$; $f''(x) > 0$ for $0 < x < 2-\sqrt{2}$ and for $x > 2+\sqrt{2}$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = (ax+b)^c x^d$ satisfies the conditions ①-⑤ if a, b, c and d are chosen as

$a = \boxed{3}$, $b = \boxed{-2}$, $c = \boxed{\frac{2}{3}}$ and $d = \boxed{-1}$.

2. Find the largest and smallest possible values of the area of the triangle cut off from the first quadrant by a line L which is tangent to the parabola $y = 15 - 2x - x^2$ at a point in the first quadrant.

[The *first quadrant* consists of the points (x, y) with $x \geq 0$ and $y \geq 0$.]

$y=0 \Rightarrow 15-2x-x^2=0 \Rightarrow x=3,\ x=-5$

Let a be the x-coordinate of the point where L is tangent to the parabola. Then $0 \leq a \leq 3$.

$y' = -2-2x \Rightarrow y'|_{x=a} = -2\cdot(a+1)$

Therefore the equation of L is: $y-(15-2a-a^2) = -2\cdot(a+1)\cdot(x-a)$

$y=0 \Rightarrow x = \dfrac{15-2a-a^2}{2\cdot(a+1)} + a = \dfrac{a^2+15}{2\cdot(a+1)}$

$x=0 \Rightarrow y = 15-2a-a^2+2\cdot(a+1)a = a^2+15$

The area of the triangle is $A = \dfrac{1}{2}\cdot\dfrac{a^2+15}{2(a+1)}\cdot(a^2+15)$.

We want to maximize/minimize $A = \dfrac{1}{4}\cdot\dfrac{(a^2+15)^2}{a+1}$ for $0 \leq a \leq 3$.

Critical points: $\dfrac{dA}{da} = \dfrac{1}{4}\cdot\left(\dfrac{2\cdot(a^2+15)\cdot 2a}{a+1} - \dfrac{(a^2+15)^2}{(a+1)^2}\right) = \dfrac{(a^2+15)\cdot(3a^2+4a-15)}{4\cdot(a+1)^2}$

$\dfrac{dA}{da} = 0 \Rightarrow 3a^2+4a-15=0 \Rightarrow a = \dfrac{5}{3}$, ~~$a=-3$~~ not in interval

$\Downarrow$

$A = \dfrac{800}{27}$

Endpoints:

$a=0 \Rightarrow A = \dfrac{225}{4}$

$a=3 \Rightarrow A = 36$

As $225 > 144 = 4\cdot 36$ and $27\cdot 36 = 972 > 800$,

$\dfrac{225}{4}$ is the largest and $\dfrac{800}{27}$ is the smallest possible value of the area of the triangle.

3. Evaluate the following integrals.

a. $\int_0^1 x^{4035}\left(x^4+1\right)^{2017}\left(3x^4+1\right)dx = \int_0^1 (x^2)^{2017}\cdot(x^4+1)^{2017}\cdot x\cdot(3x^4+1)\,dx$

$$= \int_0^1 (x^6+x^2)^{2017}\cdot(3x^5+x)\,dx = \int_0^2 u^{2017}\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{u^{2018}}{2018}\Bigg]_0^2 = \frac{2^{2016}}{1009}$$

$$\boxed{\begin{array}{l} u = x^6+x^2 \\ du = (6x^5+2x)\,dx \end{array}}$$

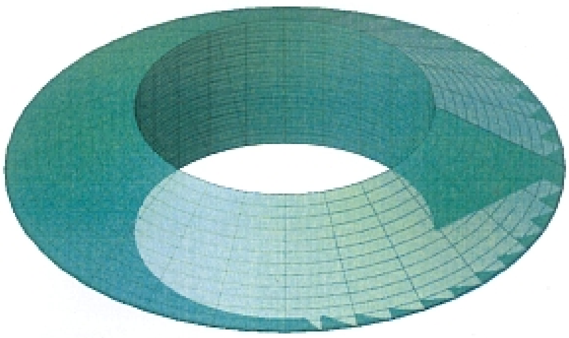
b. $\int \frac{\sin x - \cos x}{1+\sin 2x}\,dx = \int \frac{\sin x - \cos x}{\sin^2 x + \cos^2 x + 2\sin x\cos x}\,dx$

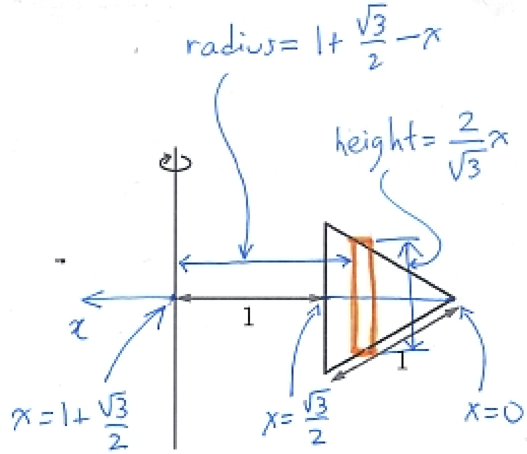
$$= \int \frac{\sin x - \cos x}{(\sin x+\cos x)^2}\,dx = -\int \frac{du}{u^2} = \frac{1}{u} + C = \frac{1}{\sin x + \cos x} + C$$

$$\boxed{\begin{array}{l} u = \sin x + \cos x \\ du = (\cos x - \sin x)\,dx \end{array}}$$

4a. A solid is generated by revolving an equilateral triangle with unit side length about a line at a unit distance from one of its sides as shown in the figure. Express the volume V of the solid as an integral using either ☐ the **washer** method or ☒ the **cylindrical shells** method by carefully defining your variable of integration, drawing a typical rectangle that generates a washer or a cylindrical shell and showing the relevant lengths and distances on the figure. [Indicate your method by ✗ing the corresponding ☐. Do not evaluate the integral!]

$$V = 2\pi \int_0^{\sqrt{3}/2} \underbrace{\left(1+\frac{\sqrt{3}}{2}-x\right)}_{\text{radius}} \cdot \underbrace{\frac{2}{\sqrt{3}}x}_{\text{height}} \, dx$$





4b. We start a rabbit farm with a pair of rabbits. Assume that at any moment the rabbit population is increasing at a rate proportional to the square of the rabbit population at that moment. Show that we will have infinitely many rabbits after a finite period of time.

Let N be the number of rabbits.

Then $\frac{dN}{dt} = k \cdot N^2$ for some positive constant k.

$$\Downarrow$$

$$\frac{dN}{N^2} = k\,dt \Rightarrow \int \frac{dN}{N^2} = \int k\,dt \Rightarrow -\frac{1}{N} = kt + C$$

Since $N(0)=2$, we have $-\frac{1}{2} = -\frac{1}{N(0)} = 0 + C \Rightarrow C = -\frac{1}{2}$

Hence $-\frac{1}{N} = kt - \frac{1}{2} \Rightarrow N = \frac{2}{1-2kt} \Rightarrow \lim\limits_{t \to \left(\frac{1}{2k}\right)^-} N = \infty$

We will have infinitely many rabbits after a time of $\frac{1}{2k}$.