Quiz # 11
Math 101-Section **01** Calculus I
11 May, 2018, Friday
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Solution Key

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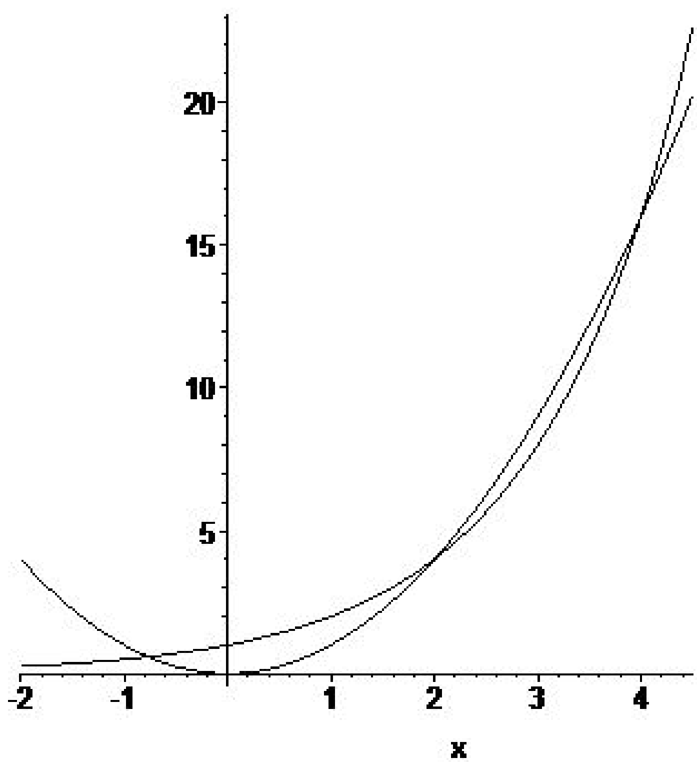
Q-1)

(i) Show that the equation $2^x = x^2$ has two positive and one negative roots.

(ii) Find the positive roots by trial and error. (*Hint: Both of the positive roots are integers.*)

(iii) Find the negative root in terms of the Lambert W function which is the inverse of the function $x \mapsto xe^x$. (*Hint: Collect all expressions involving x to one side and leave the other side as a constant. Manipulate the equation so that it looks like $f(x)e^{f(x)} = K$, where K is a constant. Then we have $W(f(x)e^{f(x)}) = W(K)$, and by the definition of the W function we have $W(f(x)e^{f(x)}) = f(x)$. Now solve $f(x) = W(K)$ as $x = f^{-1}(W(K))$.*)



Answer: The above graph shows the graphs of $y = 2^x = f(x)$ and $y = x^2 = g(x)$ together. At $x = 0$ we have $g(x) = 0 < f(x) = 1$, but as x goes to minus infinity $f(x)$ steadily decreases to zero while $g(x)$ steadily goes to plus infinity. Therefore they intersect at a point. This is the negative solution. Clearly they don't intersect at any other negative point.

When $x = 2$ we have an obvious solution. Slightly after $x = 2$, we have $x^2 > 2^x$ but 2^x grows much faster than x^2 so eventually the two graphs intersect again and from there on $2^x > x^2$. These arguments can be made precise and convincing by using some L'Hospital arguments.

The other positive root can easily be guessed as $x = 4$.

Now back to the negative root.

We have $2^x = x^2$ for some $x < 0$. Write $2^x = e^{x\ln 2}$.

$$\begin{aligned}
e^{x\ln 2} &= x^2 \\
e^{\frac{x}{2}\ln 2} &= -x, \text{ since } x < 0 \\
e^{-\frac{x}{2}\ln 2} &= -\frac{1}{x} \\
-x\, e^{-\frac{x}{2}\ln 2} &= 1 \\
(-\frac{x}{2}\ln 2)\, e^{(-\frac{x}{2}\ln 2)} &= \frac{\ln 2}{2}.
\end{aligned}$$

Now apply W function to both sides to get

$$-\frac{x}{2}\ln 2 = W(\frac{\ln 2}{2}),$$

or equivalently

$$x = -\frac{2}{\ln 2}W(\frac{\ln 2}{2}).$$

This gives

$$x \approx -0.76.$$

Note that for the positive roots we will have $x = -\frac{2}{\ln 2}W(-\frac{\ln 2}{2})$. In fact W function has many branches, like the $\arctan$ function, and its branches are indexed by integers and $2^x = x^2$ has infinitely many solutions, all except three, are complex numbers. The real roots are given by

$$-0.76... = -\frac{2}{\ln 2}W(0, \frac{\ln 2}{2}),\ 2 = -\frac{2}{\ln 2}W(0, -\frac{\ln 2}{2}),\ 4 = -\frac{2}{\ln 2}W(1, -\frac{\ln 2}{2}).$$