Quiz # 11
Math 101-Section **06** Calculus I
10 May, 2018, Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Bilkent University

Name: ....

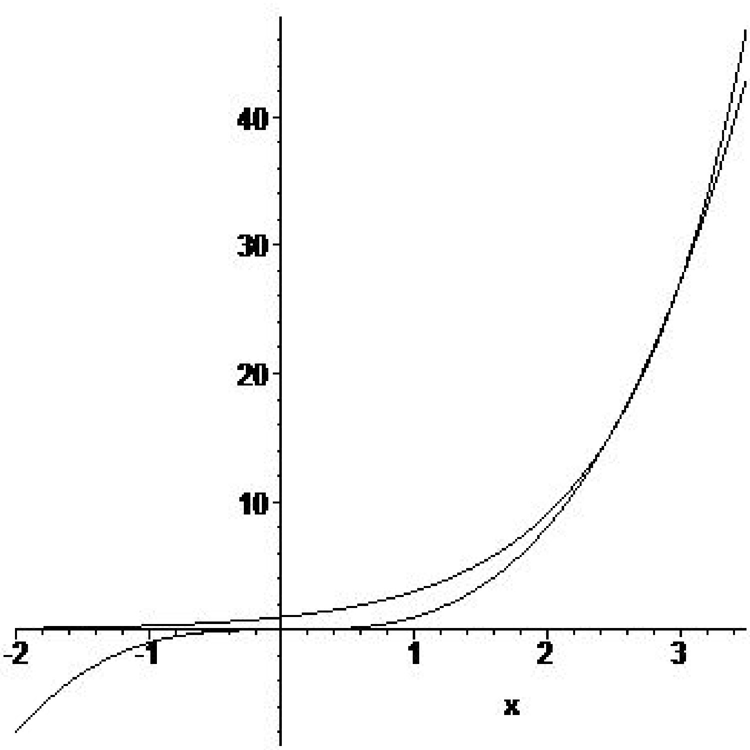
Department: .... Student ID: ....

Q-1)

(i) Show that the equation $3^x = x^3$ has two positive roots but no negative roots.

(ii) Find one of the positive roots by trial and error. (*Hint: It is an integer.*)

(iii) Find the other positive root in terms of the Lambert W function which is the inverse of the function $x \mapsto xe^x$. (*Hint: Collect all expressions involving x to one side and leave the other side as a constant. Manipulate the equation so that it looks like $f(x)e^{f(x)} = K$, where K is a constant. Then we have $W(f(x)e^{f(x)}) = W(K)$, and by the definition of the W function we have $W(f(x)e^{f(x)}) = f(x)$. Now solve $f(x) = W(K)$ as $x = f^{-1}(W(K))$.*)



Answer: The above graph shows the graphs of $y = 3^x$ and $y = x^3$ together. At $x = 0$ we have 3^x equal 1 and x^3 equal to zero. For negative x, we have that 3^x is always positive but x^3 is always negative so they don't intersect. Thus there is no negative solution.

When $x = 3$ we have an obvious solution. After $x = 3$, we have $x^3 < 3^x$, but slightly before $x = 3$ we have $x^3 > 3^x$. But at $x = 0$ this inequality is again reversed, so in between there is a solution. These arguments can be made precise and convincing by using some L'Hospital arguments.

Now to find this other root:

We have $3^x = x^3$ for some $0 < x < 3$. Write $3^x = e^{x \ln 3}$.

$$\begin{aligned} e^{x \ln 3} &= x^3 \\ e^{\frac{x}{3} \ln 3} &= x \\ e^{-\frac{x}{3} \ln 3} &= \frac{1}{x} \\ -x\, e^{-\frac{x}{3} \ln 3} &= -1 \\ (-\frac{x}{3} \ln 3)\, e^{(-\frac{x}{3} \ln 3)} &= -\frac{\ln 3}{3}. \end{aligned}$$

Now apply W function to both sides to get

$$-\frac{x}{3} \ln 3 = W(-\frac{\ln 3}{3}),$$

or equivalently

$$x = -\frac{3}{\ln 3} W(\frac{\ln 3}{3}).$$

This gives

$$x \approx 2.47.$$

In fact the W function has many branches like the $\arctan$ function and its branches are indexed by integers. Hence $3^x = x^3$ has infinitely many solutions all of which, except the two, are complex numbers. The real roots are given as

$$2.47... = -\frac{3}{\ln 3} W(0, \frac{\ln 3}{3}),\ 3 = -\frac{3}{\ln 3} W(-1, \frac{\ln 3}{3}).$$