Date: 10 March 2018, Saturday

Time: 14:00-16:00

NAME:...

STUDENT NO:.......................................

YOUR DEPARTMENT:..

CIRCLE YOUR SECTION: **1 2 3 4 5 6**

Math 101 Calculus I - First Midterm Exam - Solution Key

1	2	3	4	TOTAL
20	30	20	30	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail, unless asked otherwise. A correct answer without proper or too much reasoning may not get any credit.

The comprehensive list of *Math 101 Exam Rules* **in full detail is available on Moodle.** The following are only a few reminders:

- The exam consists of 4 questions.
- Read the questions carefully.
- **Solutions, not answers, get points.** Show all your work in well-organized mathematical sentences and explain your reasoning fully.
- **What can not be read will not be read.** Write clearly and cleanly.
- Simplify your answers as far as possible.
- Calculators and dictionaries are not allowed.
- **Turn off and leave your mobile phones with the exam proctor before the exam starts.**
- *This exam is being recorded. It is in your best interest not to give the slightest impression of doing anything improper, against the exam rules or the general rules of academic honesty.*

NAME: STUDENT NO:

Q-1) Let $f(x) = \begin{cases} 1 & x < -1 \\ x+1 & -1 \le x < 0 \\ x^2 & 0 \le x < 2 \\ 5 & x = 2 \\ x^3 - 4x^2 + 12 & x > 2 \end{cases}$

Fill in the following boxes if the required number exists; otherwise put a cross in the box. No explanation is necessary.

$\lim_{x\to -1^-} f(x) = \boxed{1}$ $\lim_{x\to -1^+} f(x) = \boxed{0}$ $\lim_{x\to 0^-} f(x) = \boxed{1}$ $\lim_{x\to 0^+} f(x) = \boxed{0}$

$\lim_{x\to 2^-} f(x) = \boxed{4}$ $\lim_{x\to 2^+} f(x) = \boxed{4}$ $\lim_{x\to 2^-} f'(x) = \boxed{4}$ $\lim_{x\to 2^+} f'(x) = \boxed{-4}$

$f'(2) = \boxed{\times}$ $f'(4) = \boxed{16}$

Grading: Each correct answer is 2 points

You can do your calculations below this line; we will not read them!

NAME: STUDENT NO:

Q-2) Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

This function is continuous everywhere. You don't have to show this. This question will show you that $f'(x)$ is defined for all real numbers, yet it is not continuous at $x = 0$. In order to see this, first

Grading: 10+10+10 points

(a) show that $f'(0) = 0$. Do this by using the limit definition of derivative.

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h}}{h} = \lim_{h\to 0} h \sin\frac{1}{h} = 0, \text{ by the Squeeze Theorem.}$$

(b) Find $f'(x)$ for $x \neq 0$ using the rules of differentiation.

$$f'(x) = [2x] \sin\frac{1}{x} + x^2 \left[-\frac{1}{x^2} \cos\frac{1}{x} \right] = 2x \sin\frac{1}{x} - \cos\frac{1}{x}.$$

(c) Show that $f'(x)$ is not continuous at $x = 0$

$f'(0)$ exists and is zero, but since $\lim\limits_{x\to 0} \cos\frac{1}{x}$ does not exit, $\lim\limits_{x\to 0} f'(x)$ does not exit, so $f'(0) \neq \lim\limits_{x\to 0} f'(x)$.

This shows that $f'(x)$ is not continuous at $x = 0$.

NAME: STUDENT NO:

Q-3 Let $p(x) = x^3 - 3x - 1$.

(i) Show that $p(x) = 0$ has two negative and one positive root.

(ii) If c is the positive root of $p(x) = 0$, find a number a such that $|a - c| < 0.5$.

(iii) Find the absolute minimum and maximum of $p(x)$ on $[-2, 0]$.

Grading: 5+5+10 points

Solution: *Show your work in reasonable detail.*

(i) $p(-2) = -3, \quad p(-1) = 1, \quad p(0) = -1, \quad p(1) = -3, \quad p(2) = 1$

Chasing sign changes and evoking the Intermediate Value Theorem, we see that there are roots in the intervals $(-2, -1)$, $(-1, 0)$, and $(1, 2)$, as expected.

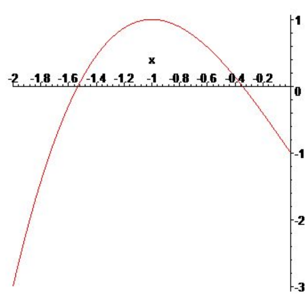
(ii) Check that $p(3/2) < 0$, so the root c is in the interval $(1.5, 2)$ Since the length of this interval is 0.5, any a in this interval satisfies the requirement.

(iii) $p'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0$; the relevant root is $x = -1$.

$p(-2)$, $p(-1)$, $p(0)$ are already calculated above.

Hence on $[-2, 0]$, the absolute minimum is -3 and occurs at the end point $x = -2$; the maximum is 1 and occurs at $x = -1$.

Here is the graph



NAME: STUDENT NO:

Q-4)

(i) Assume that $x^3 + x^2y + y^4 = 13$ defines y as a differentiable function of x. Find y'' at the point $(2,1)$.

(ii) In a right triangle, at a certain time, assume that the shorter leg is increasing at a rate of α cm/sec, and at that time the longer leg is 24 cm and is increasing at the rate of 2α cm/sec, while the hypothenuse is 25 cm and is increasing at the rate of 33 cm/sec. Find α.

Grading: 20+10 points

Solution: *Show your work in reasonable detail.*

(i) Implicit differentiation gives

$$3x^2 + 2xy + x^2y' + 4y^3y' = 0.$$

Putting $(x,y) = (2,1)$, we get

$$y' = -2.$$

Implicit differentiation again gives

$$[6x] + [2y + 2xy'] + [2xy' + x^2y''] + [12y^2(y')^2 + 4y^3y''] = 0.$$

Putting $(x,y,y') = (2,1,-2)$, we get

$$y'' = -\frac{23}{4}.$$

(ii) Let x, y, z be differentiable functions of time, and $x^2 + y^2 = z^2$ with $x < y$.

Implicit differentiation gives, after cancelling out a factor of 2

$$x\,x' + y\,y' = z\,z'$$

When $y = 24$ and $z = 25$, we get $x = 7$. Hence putting in

$$x = 7,\ x' = \alpha,\ y = 24,\ y' = 2\alpha,\ z = 25,\ z' = 33,$$

we get

$$\alpha = 15.$$