Date: 14 April 2018, Saturday

Time: **14:00-16:00**

NAME:..

STUDENT NO:..

YOUR DEPARTMENT:..

CIRCLE YOUR SECTION: **1 2 3 4 5 6**

Math 101 Calculus I - Second Midterm Exam – Solution Key

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail, unless asked otherwise. A correct answer without proper or too much reasoning may not get any credit.

The comprehensive list of *Math 101 Exam Rules* **in full detail is available on Moodle.** The following are only a few reminders:

- The exam consists of 4 questions.
- Read the questions carefully.
- **Solutions, not answers, get points.** Show all your work in well-organized mathematical sentences and explain your reasoning fully, unless asked otherwise.
- **What can not be read will not be read.** Write clearly and cleanly.
- Simplify your answers as far as possible.
- Calculators and dictionaries are not allowed.
- **Turn off and leave your mobile phones with the exam proctor before the exam starts.**
- *This exam is being recorded. It is in your best interest not to give the slightest impression of doing anything improper, against the exam rules or the general rules of academic honesty.*

Q-1) Calculus is all about derivatives! You deserve to pass only if you can correctly calculate derivatives. So, calculate the following derivatives and write the answer inside the box provided. No explanation is necessary. No partial credits are granted.

Let $f(x) = (x-1)(x+2)(x-3)(x+4)$ and $g(x) = \dfrac{x^3+x^2+3x+1}{x^2+1}$.

$(f \circ g)'(0) = \boxed{-90}$ $(g \circ f)'(-2) = \boxed{45}$

Now suppose that f and g are differentiable functions and $g(1) = 2$, $g'(1) = 3$, $g''(1) = 5$. Further assume that $f(n) = n+5$, $f'(n) = 9+n$, $f''(n) = 11+n$ whenever n is an integer. (Don't worry, such functions exist!)

$(f \circ g)''(1) = \boxed{172}$ $(g \circ f)''(-4) = \boxed{146}$

As examples of functions with the above properties we can take:

$$f(x) = 5 + x + (8+x)\frac{\sin 2\pi x}{2\pi} + (9+x)\frac{\sin^2 2\pi x}{8\pi^2}, \quad g(x) = 2 + 3(x-1) + \frac{5}{2}(x-1)^2.$$

Finally assume that y is defined as a differentiable function of x around the point $(1,2)$ by the equation $x^7 + x^3y^4 + x^4y + y^3 = 27$.

$\left(y'\Big|_{(x,y)=(1,2)} \right) = \boxed{-\frac{7}{5}}$

Grading: 5 points each

Q-2) Let $f(x) = \dfrac{x^3}{x^2-1}$. Then, $f'(x) = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$, $f''(x) = \dfrac{2x(x^2+3)}{(x^2-1)^3}$.

(a) Fill in the table below with the sign of f' and f'' and shape of f determined by these signs.

x	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	
$f''(x)$	$-$	$-$	$+$	$-$	$+$	$+$
$f'(x)$	$+$	$-$	$-$	$-$	$-$	$+$
$f(x)$						

(b) Determine local extrema of f if any.

Local min is $f(\sqrt{3}) = 3^{3/2}/2$, max is $f(-\sqrt{3}) = -f(\sqrt{3})$.

(c) Determine whether f is even, odd or neither.
It is odd since $f(-x) = -f(x)$.

(d) List the inflection point(s) if any.
Only inflection point is $(0, f(0))$ since concavity changes and f is continuous at $x = 0$.

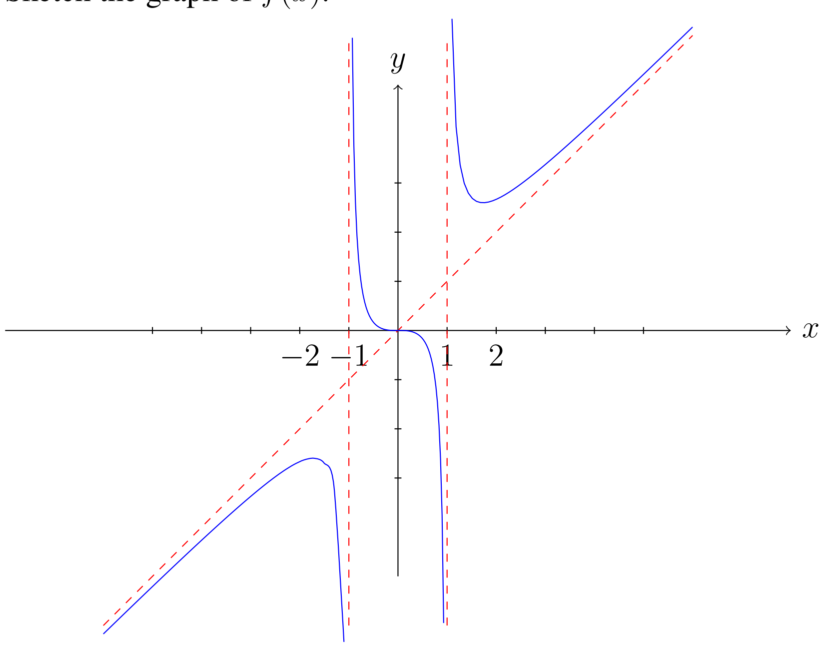
(e) Find vertical asymptote(s) inf any.
$x = \pm 1$ are vertical asympote(s) since

$$\lim_{x\to 1^+} f = \infty = \lim_{x\to -1^+} f.$$

(f) Find slant asymptote(s) if any.
$y = x$ is the slant asympote since

$$\lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \frac{x}{x^2-1} = 0.$$

(g) Sketch the graph of $f(x)$.



Q-3 Let $f(x) = x^2$ and $g(x) = x^3$.

(i) Find the area between these curves on $0 \le x \le 2$.

(ii) Find the volume obtained by revolving the above region around the x-axis.

(iii) Find the volume obtained by revolving the region between these curves around the y-axis for $0 \le y \le 1$.

Grading: 5+10+10 points

Solution: *Show your work in reasonable detail.*

(i)

$$\text{Area} = \int_0^1 (x^2 - x^3)\,dx + \int_1^2 (x^3 - x^2)\,dx = \frac{3}{2}.$$

(ii)

$$\text{Volume} = \pi \int_0^1 (x^4 - x^6)\,dx + \pi \int_1^2 (x^6 - x^4)\,dx = 12\pi.$$

(iii)

$$\text{Volume} = \pi \int_0^1 (y^{2/3} - y)\,dy = \frac{\pi}{10}.$$

Q-4)

(i) Evaluate the indefinite integral $\int \sin x \, \sin(\cos x)\, dx$.

(ii) Evaluate $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{(1+\tan x)^{1/7}}\, dx$.

(iii) Evaluate the limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k(2k+n)-n^2}{n^3}$, by first converting it into a definite integral.

Grading: 5+10+10 points

Solution: *Show your work in reasonable detail.*

(i) Put $u = \cos x$, then

$$\int \sin x \, \sin(\cos x)\, dx = -\int \sin u \, du = \cos u + C = \cos(\cos x) + C.$$

(ii) Put $u = 1 + \tan x$, then

$$\int_0^{\pi/4} \frac{\sec^2 x}{(1+\tan x)^{1/7}}\, dx = \int_1^2 u^{-1/7}\, du = \frac{7}{6}\left(u^{6/7}\Big|_1^2 \right) = \frac{7}{6}(2^{6/7} - 1) \approx 0.95.$$

(iii) First write $\displaystyle\frac{k(2k+n)-n^2}{n^3} = \frac{k(2k+n)-n^2}{n^2}\,\frac{1}{n}$.

Then $\displaystyle\frac{k(2k+n)-n^2}{n^2} = 2\left(\frac{k}{n}\right)^2 + \left(\frac{k}{n}\right) - 1.$

So we take $f(x) = 2x^2 + x - 1$ on $[0,1]$, and the required limit is then equal to

$$\int_0^1 (2x^2 + x - 1)\, dx = \frac{1}{6}.$$