Quiz # 5
Math 101-Section **13** Calculus I
8 November 2018, Thursday
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Solution Key

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Q-1) Let $f(x) = \dfrac{x}{2} + x^2 \sin \dfrac{1}{x}$ for $x \neq 0$, and $f(0) = 0$.

1. Calculate $f'(0)$, and check that it is positive.
2. Show however that f is not increasing on any interval of the form $(0, \epsilon)$ where $\epsilon > 0$.

Solution:

1. Note that since $f(0)$ is defined as 0 at $x = 0$, we assume that $g(x) = x^2 \sin \frac{1}{x}$ is defined as 0 at $x = 0$.

Note also that

$$-x \leq x \sin \frac{1}{x} \leq x, \quad \text{for } x \neq 0,$$

so $\lim\limits_{x\to 0} x \sin \dfrac{1}{x} = 0$ by the Sandwich Theorem. Hence

$$\left.\frac{d}{dx}\right|_{x=0} x^2 \sin \frac{1}{x} = \lim_{x\to 0} \frac{g(x) - g(0)}{x} = \lim_{x\to 0} x \sin \frac{1}{x} = 0.$$

This means

$$f'(0) = \left.\frac{d}{dx}\right|_{x=0} \frac{x}{2} + \left.\frac{d}{dx}\right|_{x=0} x^2 \sin \frac{1}{x} = \frac{1}{2}, \text{ and hence is positive.}$$

2. Choose any $\epsilon > 0$ you like. Let N be a large integer such that

$$\frac{1}{2n\pi}, \frac{1}{\frac{\pi}{2} + 2n\pi} \in (0, \epsilon), \quad \text{for all } n \geq N.$$

Note that

$$f'(x) = \frac{1}{2} + 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad \text{for } x > 0.$$

Then it is easy to see that

$$f'\left(\frac{1}{2n\pi}\right) < 0 \quad \text{and } f'\left(\frac{1}{\frac{\pi}{2} + 2n\pi}\right) > 0, \quad \text{for all } n \geq N.$$

This means that f is not increasing on the interval $(0, \epsilon)$.