Bilkent University

Quiz # 11
Math 101-Section 08 Calculus I
19 December 2019, Thursday
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Solution Key

Q-1) Let $f(x) = x^{1/x}$, where $x > 0$.

(a) Find $f'(x)$.

(b) Find $f''(x)$.

(c) Calculate $\lim_{x \to 0^+} f(x)$.

(d) Calculate $\lim_{x \to \infty} f(x)$.

(e) Find the minimum and maximum values of $f(x)$ for $x > 0$.

Solution:

(a) $f(x) = x^{1/x} = e^{(\ln x)/x}$, therefore, by the chain rule,

$$f'(x) = e^{(\ln x)/x} \frac{d}{dx} \frac{\ln x}{x} = x^{1/x} \left(\frac{1 - \ln x}{x^2} \right).$$

(b) Now we apply the product rule together with the chain rule to obtain

$$f''(x) = x^{1/x} \left(\frac{1 - \ln x}{x^2} \right)^2 + x^{1/x} \left(\frac{2 \ln x - 3}{x^3} \right).$$

(c) $\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$, therefore $\lim_{x \to 0^+} x^{1/x} = \lim_{x \to 0^+} e^{(\ln x)/x} = 0$.

(d) $\lim_{x \to \infty} \frac{\ln x}{x} = 0$ by L'Hospital's rule, therefore $\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{(\ln x)/x} = 1$.

(e) $f'(x) = 0$ when $x = e$, and $f(e) = e^{1/e} \approx 1.44$. Hence on $(0, \infty)$, f has no minimum but its maximum is $f(e)$. Note that 0 is not a minimum for f since f never takes the value 0.

Here is a graph of $y = f(x)$ for your information, not required as part of this quiz.