Bilkent University

Quiz # 06
Math 101-Section 12 Calculus I
22 November 2020 Sunday
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Solution Key

For this problem you need to use a computer algebra software. If none is installed on your computer, use https://www.wolframalpha.com/

You may also want to take a look at
`https://en.wikipedia.org/wiki/Leibniz_integral_rule#Variable_limits_form`

Q-1) Let

$$f(x) = \int_{x^3}^{x^2} \sqrt{1+t^4}\, dt, \quad \text{for } x \in [0,1].$$

Clearly $f(x) \geq 0$ and $f(0) = f(1) = 0$. Find the maximum value of $f(x)$ on $[0,1]$.

Solution:

Recall the Leibniz rule:

$$\frac{d}{dx}\int_{h(x)}^{g(x)} F(t)\, dt = F(g(x))\, g'(x) - F(h(x))\, h'(x).$$

Thus we have

$$f'(x) = \sqrt{1+x^8}\, 2x - \sqrt{1+x^{12}}\, 3x^2.$$

Here $x = 0$ is a solution but it is an end point. Cancelling out x and equating $f'(x)$ to zero we get

$$2\sqrt{1+x^8} = 3x\sqrt{1+x^{12}}.$$

Squaring both sides we get

$$4 + 4x^8 = 9x^2 + 9x^{14}.$$

There is only one root of this on $[0,1]$, which is $x_0 = 0.6782...$. Since the end points will not contribute to the maximum value, this critical point gives the maximum value of f as

$$f_{max} = f(x_0) = 0.1497...$$

Here is the graph of $y = f(x)$, not reqired as part of this quiz.