1. Evaluate the following limits. [Do not use L'Hôpital's Rule!]

a. $\lim\limits_{x\to-\infty} \dfrac{\sin(1/x)}{\sqrt{x^2+1}+x} = \lim\limits_{x\to-\infty} \dfrac{\sin(1/x)\cdot(\sqrt{x^2+1}-x)}{(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x)}$

$= \lim\limits_{x\to-\infty} \left(\sin(1/x)\cdot\left(|x|\cdot\sqrt{1+\frac{1}{x^2}}-x\right)\right)$

$= \lim\limits_{x\to-\infty} \left(\sin(1/x)\cdot\left(-x\sqrt{1+\frac{1}{x^2}}-x\right)\right)$

$= -\underbrace{\lim\limits_{x\to-\infty} \dfrac{\sin(1/x)}{1/x}}_{1} \cdot \lim\limits_{x\to-\infty}\left(\sqrt{1+\frac{1}{x^2}}+1\right) = -1\cdot(1+1) = -2$

b. $\lim\limits_{x\to\pi/2} \dfrac{1+\cos 2x}{1-\sqrt[3]{\sin x}} = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{2\cos^2 x\cdot(1+(\sin x)^{1/3}+(\sin x)^{2/3})}{(1-(\sin x)^{1/3})\cdot(1+(\sin x)^{1/3}+(\sin x)^{2/3})}$

$= 2\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1-\sin^2 x}{1-\sin x}\cdot\lim\limits_{x\to\frac{\pi}{2}}(1+(\sin x)^{1/3}+(\sin x)^{2/3})$

$= 2\cdot\lim\limits_{x\to\frac{\pi}{2}}(1+\sin x)\cdot(1+1+1) = 2\cdot(1+1)\cdot 3 = 12$

2. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(\pi/6,\pi/3)}$ if y is a differentiable function of x satisfying the equation:

$$y\sin(2y-x)=2x$$

$$\Downarrow \ d/dx$$

$$y'\sin(2y-x)+y\cos(2y-x)\cdot(2y'-1)=2$$

$$\Downarrow \leftarrow \boxed{(x,y)=\left(\tfrac{\pi}{6},\tfrac{\pi}{3}\right)}$$

$$y'\underbrace{\sin\tfrac{\pi}{2}}_{1}+\tfrac{\pi}{3}\underbrace{\cos\tfrac{\pi}{2}}_{0}\cdot(2y'-1)=2$$

$$\Downarrow$$

$$y'=2 \text{ at } (x,y)=\left(\tfrac{\pi}{6},\tfrac{\pi}{3}\right)$$

$$\Downarrow \ d/dx$$

$$y''\sin(2y-x)+y'\cos(2y-x)\cdot(2y'-1)$$
$$+\,y'\cos(2y-x)\cdot(2y'-1)+y\cdot(-\sin(2y-x))\cdot(2y'-1)^2$$
$$+\,y\cos(2y-x)\cdot 2y''=0$$

$$\Downarrow \leftarrow \boxed{(x,y)=\left(\tfrac{\pi}{6},\tfrac{\pi}{3}\right),\ y'=2}$$

$$y''\underbrace{\sin\tfrac{\pi}{2}}_{1}+2\underbrace{\cos\tfrac{\pi}{2}}_{0}\cdot(2\cdot2-1)+2\underbrace{\cos\tfrac{\pi}{2}}_{0}\cdot(2\cdot2-1)+\tfrac{\pi}{3}\cdot\Big(-\underbrace{\sin\tfrac{\pi}{2}}_{1}\Big)\cdot(2\cdot2-1)^2+\tfrac{\pi}{3}\underbrace{\cos\tfrac{\pi}{2}}_{0}\cdot 2y''=0$$

$$\Downarrow$$

$$y''=3\pi \text{ at } (x,y)=\left(\tfrac{\pi}{6},\tfrac{\pi}{3}\right)$$

3. The points P and Q are moving along the graph of a twice-differentiable function $y = f(x)$ in the xy-plane in such a way that their coordinates are differentiable functions of time t, and the tangent line to the graph at the point P intersects the graph also at the point Q at all times. (Assume that the coordinates are measured in meters and the time is measured in seconds.)

Find $f''(2)$ if

① the x-coordinate of Q is -1 and decreasing at a rate of 3 m/s when the x-coordinate of P is 2 and increasing at a rate of 4 m/s,

② $y = 9x - 8$ is an equation for the tangent line to the graph of f at the point with $x = 2$, and

③ $y = -6x - 23$ is an equation for the tangent line to the graph of f at the point with $x = -1$.

Let a and b be the x-coordinates of P and Q, respectively. Then:

$$f(b) = f(a) + f'(a)\cdot(b-a) \quad \text{at all times}$$

$$\Big\Downarrow \; d/dt$$

$$f'(b)\frac{db}{dt} = f'(a)\frac{da}{dt} + f''(a)\frac{da}{dt}\cdot(b-a) + f'(a)\cdot\left(\frac{db}{dt} - \frac{da}{dt}\right)$$

$$\Big\Downarrow$$

$$f''(a) = \frac{f'(b) - f'(a)}{b-a}\cdot\frac{db/dt}{da/dt}$$

$$\boxed{a=2,\ \frac{da}{dt}=4,\ b=-1,\ \frac{db}{dt}=-3 \text{ by ①} \quad f'(2)=9 \text{ by ②},\ f'(-1)=-6 \text{ by ③}}$$

$$\Big\Downarrow$$

$$f''(2) = \frac{-6-9}{-1-2}\cdot\frac{-3}{4} = -\frac{15}{4}$$

4. In each of the following, if the given statement is true, then mark the ☐ to the left of TRUE with a ✗ and prove the statement; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

a. If f is differentiable on $(0,\infty)$ and $f(1/x)=f(x)$ for all $x>0$, then there is a c in $(0,\infty)$ such that $f'(c)=0$.

☒ TRUE ☐ FALSE

$$f(1/x)=f(x) \Rightarrow f'(1/x)\cdot(-1/x^2)=f'(x) \xRightarrow{\boxed{x=1}} -f'(1)=f'(1) \Rightarrow f'(1)=0$$

b. If f is differentiable on $(0,\infty)$ and $f(x^2)=(f(x))^3$ for all $x>0$, then there is a c in $(0,\infty)$ such that $f'(c)=0$.

☒ TRUE ☐ FALSE

$$f(x^2)=f(x)^3 \xRightarrow{\boxed{x=1}} f(1)=f(1)^3 \Rightarrow f(1) \text{ is } 1, 0 \text{ or } -1$$

$$\Downarrow$$

$$f'(x^2)\cdot 2x=3f(x)^2\cdot f'(x) \xRightarrow{\boxed{x=1}} 2f'(1)=3f(1)^2 f'(1) \Rightarrow (3f(1)^2-2)\cdot f'(1)=0 \Rightarrow f'(1)=0$$

c. If f is differentiable on $(0,\infty)$ and $f(x)f(2x)\geq 0$ for all $x>0$, then there is a c in $(0,\infty)$ such that $f'(c)=0$.

☐ TRUE ☒ FALSE

Let $f(x)=x$.

Then $f(x)f(2x)=x\cdot 2x=2x^2\geq 0$ for all $x>0$,

but $f'(x)=1\neq 0$ for all x.

d. If f is differentiable on $(0,\infty)$ and $f(x)f(2x)\leq 0$ for all $x>0$, then there is a c in $(0,\infty)$ such that $f'(c)=0$.

☒ TRUE ☐ FALSE

f is diff'ble on $(0,\infty) \Rightarrow f$ is continuous on $(0,\infty)$.

- If $f(1)f(2)<0$, then applying IVT to f on $[1,2]$ we conclude that f has a zero in $(1,2)$
- Otherwise, $f(1)f(2)=0$ and f has a zero at 1 or 2.

Therefore, in both cases there is a in $[1,2]$ such that $f(a)=0$.

Similarly, there is b in $[3,6]$ such that $f(b)=0$.

Applying Rolle's Theorem to f on $[a,b]$ we conclude that there is c in (a,b) such that $f'(c)=0$.

5. A twice-differentiable function f on $(-\infty, \infty)$ satisfies the following conditions:

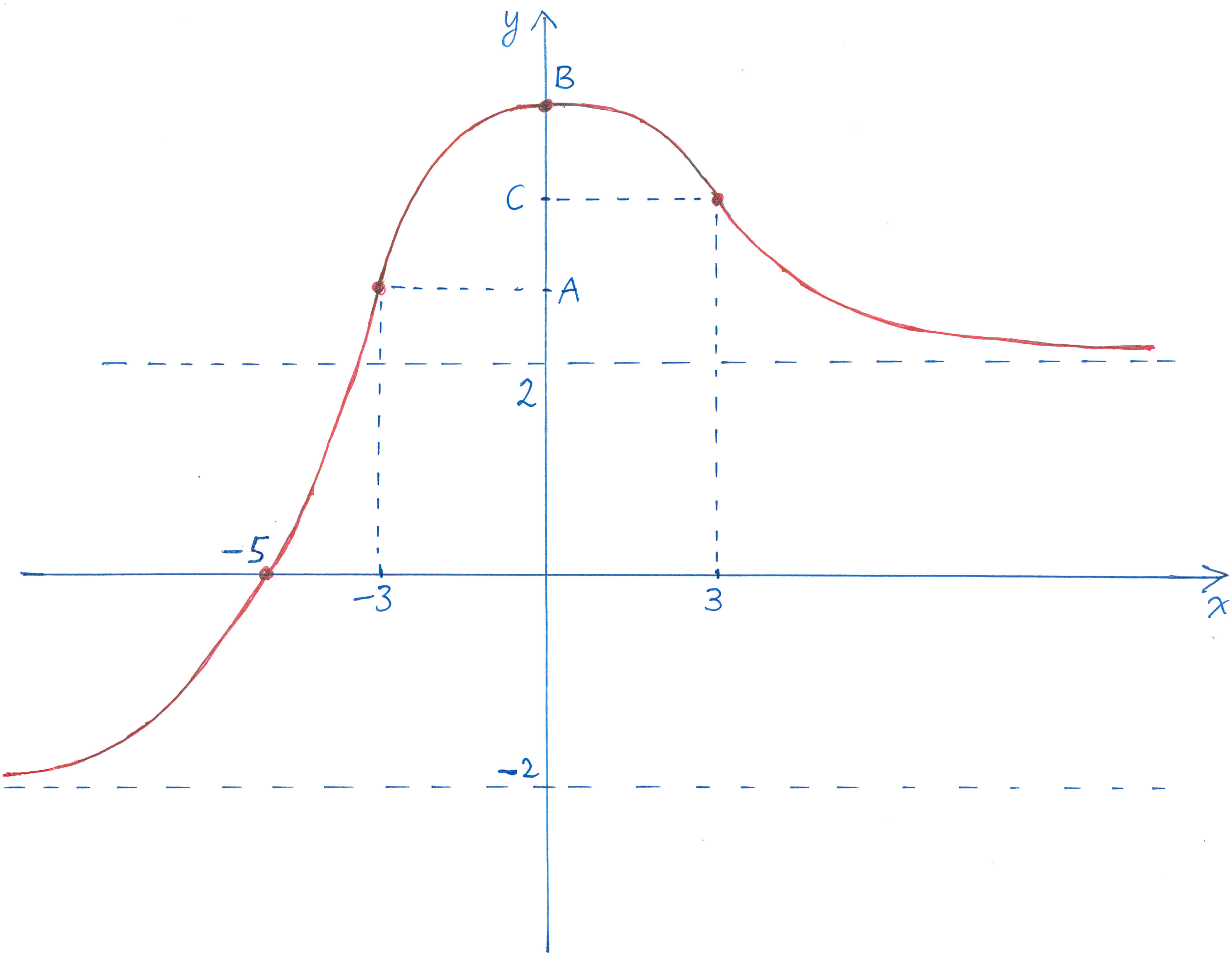
① $f(-5) = 0$, $f(-3) = A$, $f(0) = B$, $f(3) = C$, where A, B, C are real numbers such that $2 < A < C$

② $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 2$

③ $f'(x) > 0$ for $x < 0$, $f'(x) < 0$ for $x > 0$

④ $f''(0) = 0$, $f''(x) > 0$ for $x < -3$ and for $x > 3$, $f''(x) < 0$ for $-3 < x < 0$ and for $0 < x < 3$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = \dfrac{ax^3 + b}{|x|^3 + c}$ satisfies the conditions ①-④ if a, b and c are chosen as

$a = \boxed{2}$, $b = \boxed{250}$ and $c = \boxed{54}$.