1. Find all values of the constant a for which the limit

$$\lim_{x\to 0} \frac{xe^{ax^2} - \sin x}{x^5}$$

exists and evaluate the limit for each of these values of a.

$$\lim_{x\to 0} \frac{xe^{ax^2} - \sin x}{x^5} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{e^{ax^2} + x\cdot 2ax\, e^{ax^2} - \cos x}{5x^4}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{2ax e^{ax^2} + 4ax e^{ax^2} + 2ax^2\cdot 2ax\, e^{ax^2} + \sin x}{20x^3}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{6a e^{ax^2} + 6ax\cdot 2ax\, e^{ax^2} + 12a^2x^2 e^{ax^2} + 4a^2x^3\cdot 2ax e^{ax^2} + \cos x}{60x^2}$$

This limit does not exist unless $6a+1=0$. So, $a=-\frac{1}{6}$ from here on:

$$= \lim_{x\to 0} \frac{-e^{-x^2/6} + \frac{2}{3}x^2 e^{-x^2/6} - \frac{1}{27}x^4 e^{-x^2/6} + \cos x}{60x^2}$$

$$\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{x}{3} e^{-x^2/6} + \frac{4}{3}x e^{-x^2/6} - \frac{2}{9}x^3 e^{-x^2/6} - \frac{4}{27}x^3 e^{-x^2/6} + \frac{1}{81}x^5 e^{-x^2/6} - \sin x}{120x}$$

$$= \frac{1}{120}\lim_{x\to 0}\left(\frac{5}{3}e^{-x^2/6} - \frac{10}{27}x^2 e^{-x^2/6} + \frac{1}{81}x^4 e^{-x^2/6} - \frac{\sin x}{x}\right)$$

$$= \frac{1}{120}\cdot\left(\frac{5}{3} - 0 + 0 - 1\right) = \frac{1}{180}$$

2a. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{(x+1)(2x+3)}$.

$$\int \frac{dx}{(x+1)(2x+3)} = \int \left(\frac{1}{x+1} - \frac{2}{2x+3}\right)dx = \ln|x+1| - \ln|2x+3| + C$$

$$\int_0^\infty \frac{dx}{(x+1)(2x+3)} = \lim_{c\to\infty} \int_0^c \frac{dx}{(x+1)(2x+3)} = \lim_{c\to\infty} \left[\ln\left|\frac{x+1}{2x+3}\right|\right]_0^c$$

$$= \lim_{c\to\infty} \left(\ln\left(\frac{c+1}{2c+3}\right) - \ln\left(\frac{1}{3}\right)\right) = \ln\left(\frac{1}{2}\right) - \ln\left(\frac{1}{3}\right) = \ln\left(\frac{3}{2}\right)$$

2b. Let $A = \displaystyle\int_1^{\sqrt{3}} \frac{\ln x}{x^2+1}\,dx$ and $B = \displaystyle\int_2^{2\sqrt{3}} \frac{\ln x}{x^2+4}\,dx$. Express B in terms of A.

$$B = \int_2^{2\sqrt{3}} \frac{\ln x}{x^2+4}\,dx = \int_1^{\sqrt{3}} \frac{\ln(2u)}{4u^2+4}\cdot 2\,du = \frac{1}{2}\int_1^{\sqrt{3}} \frac{\ln 2 + \ln u}{u^2+1}\,du$$

$$\boxed{\begin{aligned} x &= 2u \\ dx &= 2\,du \end{aligned}}$$

$$= \frac{\ln 2}{2}\int_1^{\sqrt{3}} \frac{du}{u^2+1} + \frac{A}{2} = \frac{\ln 2}{2}\Big[\arctan u\Big]_1^{\sqrt{3}} + \frac{A}{2}$$

$$= \frac{\ln 2}{2}\cdot\Big(\underbrace{\arctan\sqrt{3}}_{\frac{\pi}{3}} - \underbrace{\arctan 1}_{\frac{\pi}{4}}\Big) + \frac{A}{2} = \frac{\pi \ln 2}{24} + \frac{A}{2}$$

3. A function f with a continuous second derivative satisfies:

$$\int_0^{\pi/2} \sin(x) f(x)\, dx \overset{①}{=} 1 \qquad \int_0^{\pi/2} \sin(x) f'(x)\, dx \overset{③}{=} 3 \qquad \int_0^{\pi/2} \sin(2x) f(x)\, dx \overset{⑤}{=} 5$$

$$\int_0^{\pi/2} \cos(x) f(x)\, dx \overset{②}{=} 2 \qquad \int_0^{\pi/2} \cos(x) f'(x)\, dx \overset{④}{=} 4 \qquad \int_0^{\pi/2} \cos(2x) f(x)\, dx \overset{⑥}{=} 6$$

Evaluate $\displaystyle\int_0^{\pi/2} \sin(2x) f''(x)\, dx$.

$$\int_0^{\pi/2} \sin(2x) f''(x)dx = \int_0^{\pi/2} \sin(2x)\, d(f'(x)) = \Big[\sin(2x) f'(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} f'(x)\, d(\sin(2x))$$

$$= \underbrace{\sin(\pi)}_{0} f'(\tfrac{\pi}{2}) - \underbrace{\sin(0)}_{0} f'(0) - 2\int_0^{\pi/2} \cos(2x) f'(x)\, dx = -2\int_0^{\pi/2} \cos(2x)\, d(f(x))$$

$$= -2\Big[\cos(2x) f(x)\Big]_0^{\pi/2} + 2\int_0^{\pi/2} f(x)\, d(\cos(2x))$$

$$= -2 \underbrace{\cos(\pi)}_{-1} \underbrace{f(\tfrac{\pi}{2})}_{5} + 2 \underbrace{\cos(0)}_{1} \underbrace{f(0)}_{-3} - 4\underbrace{\int_0^{\pi/2} \sin(2x) f(x)\, dx}_{5 \text{ by } ⑤} = 10 - 6 - 20 = -16$$

because ③ + ② gives:

$$5 = 3 + 2 = \int_0^{\pi/2} (\sin(x) f'(x) + \cos(x) f(x))\, dx = \int_0^{\pi/2} d(\sin(x) f(x))$$

$$= \Big[\sin(x) f(x)\Big]_0^{\pi/2} = \underbrace{\sin(\tfrac{\pi}{2})}_{1} f(\tfrac{\pi}{2}) - \underbrace{\sin(0)}_{0} f(0) = f(\tfrac{\pi}{2})$$

and ④ − ① gives:

$$3 = 4 - 1 = \int_0^{\pi/2} (\cos(x) f'(x) - \sin(x) f(x))\, dx = \int_0^{\pi/2} d(\cos(x) f(x))$$

$$= \Big[\cos(x) f(x)\Big]_0^{\pi/2} = \underbrace{\cos(\tfrac{\pi}{2})}_{0} f(\tfrac{\pi}{2}) - \underbrace{\cos(0)}_{1} f(0) = -f(0)$$

4. In each of the following, if there exists a function f that satisfies the given conditions, give an example of such a function; otherwise, just write DOES NOT EXIST inside the box. No explanation is required. No partial points will be given.

a. f is continuous on $(-\infty, \infty)$ and f does not have an antiderivative on $(-\infty, \infty)$.

$f(x) = $ DNE

b. f is positive and differentiable on $(-\infty, \infty)$ and $\displaystyle\int \frac{dx}{f(x)} \neq \ln(f(x)) + C$.

$f(x) = x^2+1$

c. f is continuous on $[0, \pi]$ and $\displaystyle\int_0^\pi |f(x)|\,dx \neq \left|\int_0^\pi f(x)\,dx\right|$.

$f(x) = \cos x$

d. f is differentiable on $(0, \infty)$, $\lim_{x\to\infty} f'(x) = 0$, and $\lim_{x\to\infty} f(x)$ does not exist.

$f(x) = \sqrt{x}$

e. f is differentiable on $(0, \infty)$, $\lim_{x\to\infty} f(x) = 0$, and $\lim_{x\to\infty} f'(x)$ does not exist.

$f(x) = \dfrac{\sin(x^2)}{x}$