1a. Evaluate the limit $\lim_{x\to 1} \dfrac{x^2+2-\dfrac{12}{x+3}}{x^2+3-\dfrac{12}{x+2}}$.

(Do not use L'Hôpital's Rule!)

$$\lim_{x\to 1} \frac{x^2+2-\frac{12}{x+3}}{x^2+3-\frac{12}{x+2}} = \lim_{x\to 1}\left(\frac{(x^2+2)(x+3)-12}{(x^2+3)\cdot(x+2)-12}\cdot\frac{x+2}{x+3}\right)$$

$$= \lim_{x\to 1} \frac{x^3+3x^2+2x-6}{x^3+2x^2+3x-6}\cdot\frac{1+2}{1+3} = \lim_{x\to 1}\frac{\cancel{(x-1)}\cdot(x^2+4x+6)}{\cancel{(x-1)}\cdot(x^2+3x+6)}\cdot\frac{3}{4}$$

$$= \frac{1^2+4\cdot 1+6}{1^2+3\cdot 1+6}\cdot\frac{3}{4} = \frac{11}{10}\cdot\frac{3}{4} = \frac{33}{40}$$

1b. Suppose that a differentiable function f satisfies

$$f'(x)+3f(x^2)=f(x)^2 \qquad (*)$$

for all $x>0$ and $f(1)=5$. Find $f''(1)$.

$$(*) \xRightarrow{x=1} f'(1)+3f(1)=f(1)^2 \xRightarrow{f(1)=5} f'(1)+3\cdot 5=5^2 \Rightarrow f'(1)=10$$

$$(*) \xRightarrow{d/dx} f''(x)+3f'(x^2)\cdot 2x = 2f(x)f'(x)$$

$$\Downarrow \leftarrow x=1$$

$$f''(1)+6f'(1)=2f(1)f'(1)$$

$$\Downarrow \leftarrow f(1)=5,\ f'(1)=10$$

$$f''(1)+6\cdot 10 = 2\cdot 5\cdot 10$$

$$\Downarrow$$

$$f''(1)=40$$

2a. Find an equation for the tangent line to the graph of $y = \tan^3\left(\frac{\pi}{x}\right)$ at the point with $x = 3$.

$$y' = 3\tan^2\left(\frac{\pi}{x}\right)\sec^2\left(\frac{\pi}{x}\right)\cdot\left(-\frac{\pi}{x^2}\right)$$

$$y'\big|_{x=3} = 3\tan^2\left(\frac{\pi}{3}\right)\sec^2\left(\frac{\pi}{3}\right)\cdot\left(-\frac{\pi}{9}\right) = 3\cdot\sqrt{3}^2\cdot 2^2\cdot\left(-\frac{\pi}{9}\right) = -4\pi$$

$$y\big|_{x=3} = \tan^3\left(\frac{\pi}{3}\right) = \sqrt{3}^3 = 3\sqrt{3}$$

An equation for the tangent line is:

$$y - 3\sqrt{3} = -4\pi\cdot(x-3)$$

2b. Suppose that a differentiable function f satisfies

$$-||x|-2| < f(x) < |x| \qquad \circledast$$

for all x. Show that there is c such that $f'(c) = 0$.

Since f is differentiable, f is continuous.

$$\circledast \overset{x=0}{\Longrightarrow} -2 < f(0) < 0$$

$$\circledast \overset{x=-2}{\Longrightarrow} 0 < f(-2) < 2$$

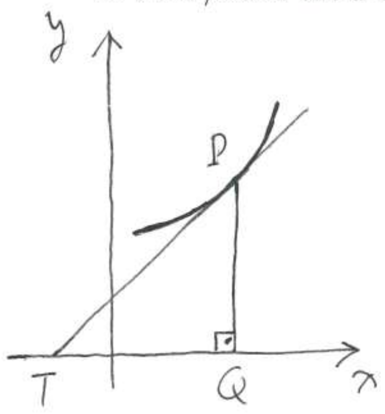
Since f is continuous on $[-2,0]$ and $f(0) < 0 < f(-2)$, by IVT there is a in $(-2,0)$ such that $f(a)=0$.

Similarly, there is b in $(0,2)$ such that $f(b)=0$.

Since f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a)=0=f(b)$, by Rolle's Theorem there is c in (a,b) such that $f'(c)=0$.

3. For a point P on the curve $y = x^2 + 5$ different from $(0, 5)$, let T and Q be points on the x-axis such that the line PT is tangent to the curve at P and PQ is perpendicular to the x-axis. Let A be the area of the triangle PQT. Assume that the coordinates are measured in centimeters and the time is measured in seconds.

Find all possible values of the x-coordinate of the point P at a moment when A is decreasing at a rate of 6 cm^2/s and the x-coordinate of the point P is increasing at a rate of 2 cm/s.



$$\tan(\hat{PTQ}) = \frac{|PQ|}{|TQ|} \Rightarrow |y'| = \frac{|y|}{|TQ|} \Rightarrow |TQ| = \frac{|y|}{|y'|}$$

$$A = \frac{1}{2}\cdot|TQ|\cdot|PQ| = \frac{1}{2}\cdot\frac{|y|}{|y'|}\cdot|y| = \frac{y^2}{2|y'|} = \frac{(x^2+5)^2}{2\cdot|2x|} = \frac{(x^2+5)^2}{4|x|}$$

$$\Rightarrow A = \begin{cases} \frac{1}{4}\left(x^3 + 10x + \frac{25}{x}\right) & \text{if } x>0 \\ -\frac{1}{4}\left(x^3 + 10x + \frac{25}{x}\right) & \text{if } x<0 \end{cases} \Rightarrow \frac{dA}{dx} = \begin{cases} \frac{1}{4}\left(3x^2 + 10 - \frac{25}{x^2}\right) & \text{if } x>0 \\ -\frac{1}{4}\left(3x^2 + 10 - \frac{25}{x^2}\right) & \text{if } x<0 \end{cases}$$

$$\frac{dA}{dt} = \frac{dA}{dx}\cdot\frac{dx}{dt} \xrightarrow{\text{at our moment}} -6 = \frac{dA}{dx}\cdot 2 \Rightarrow \frac{dA}{dx} = -3$$

Therefore:

If $x>0$, then $\frac{1}{4}\left(3x^2+10-\frac{25}{x^2}\right) = -3 \Rightarrow \underbrace{3x^4+22x^2-25}_{(x^2-1)(3x^2+25)}=0 \Rightarrow x = 1$

If $x<0$, then $-\frac{1}{4}\left(3x^2+10-\frac{25}{x^2}\right) = -3 \Rightarrow 3x^4-2x^2-25=0 \Rightarrow x^2 = \frac{2\pm\sqrt{304}}{6}$

$$\Downarrow$$

$$x = -\sqrt{\frac{1+2\sqrt{19}}{3}}$$

The possible values of x are 1 and $-\sqrt{\frac{1+2\sqrt{19}}{3}}$.

4. A function f, which is continuous on $[0, \infty)$ and twice-differentiable on $(0, \infty)$, satisfies the following conditions:

① $f(0) = 2\,,\ f(3-\sqrt{5}) = 1+\sqrt{5}\,,\ f(2) = 3$

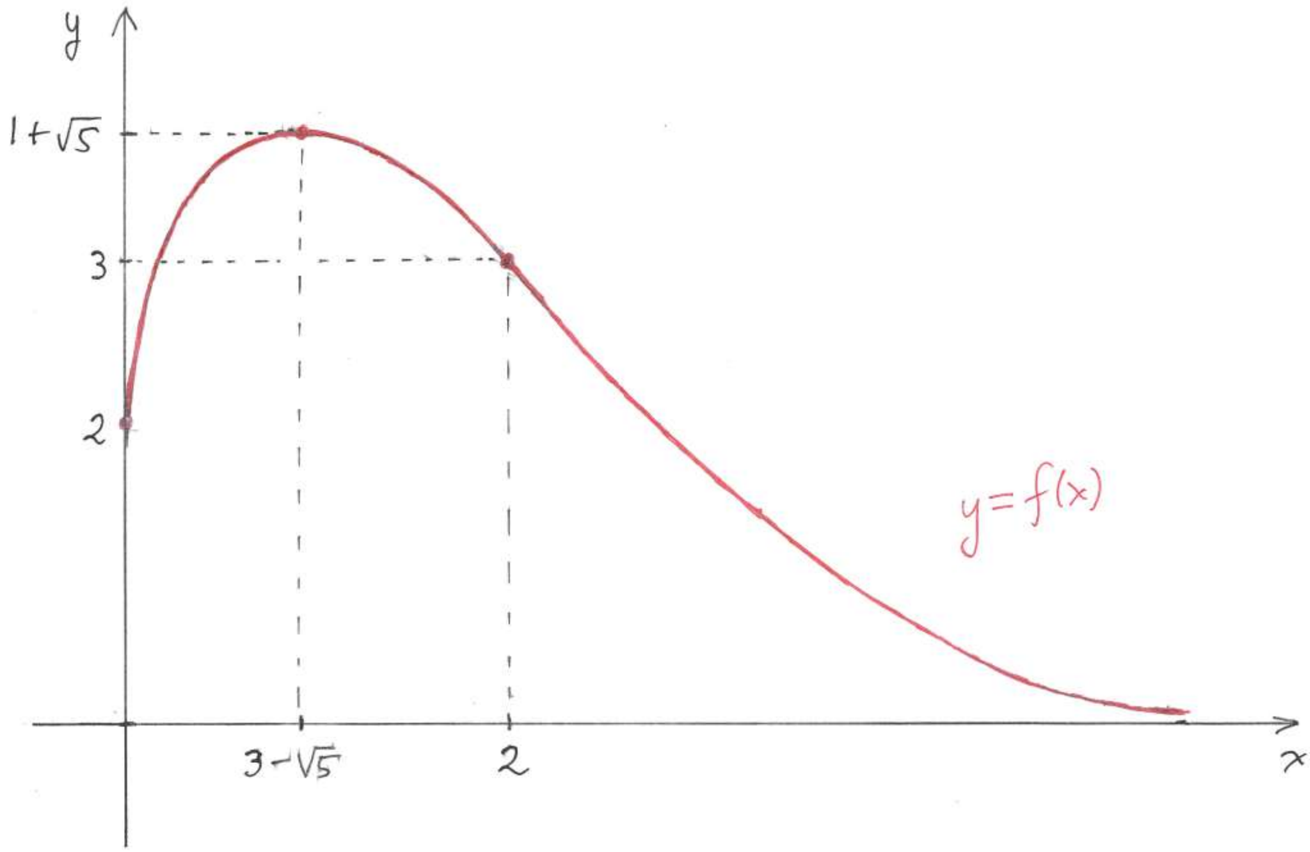
② $\lim_{x\to\infty} f(x) = 0$

③ $f'(x) > 0$ for $0 < x < 3-\sqrt{5}$, and $f'(x) < 0$ for $x > 3-\sqrt{5}$

④ $\lim_{x\to 0^+} f'(x) = \infty$

⑤ $f''(x) < 0$ for $0 < x < 2$, and $f''(x) > 0$ for $x > 2$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = \dfrac{a\sqrt{x}+b}{x+c}$ satisfies the conditions ①-⑤ at all points in its domain if a, b and c are chosen as

$a = \boxed{4\sqrt{2}}$, $b = \boxed{4}$ and $c = \boxed{2}$.