Bilkent University

Quiz # 09
Math 101-Section 08 Calculus I
9 December 2022 Friday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) (i) If $f(x) = x^{x^x}$, calculate $\left.\dfrac{df}{dx}\right|_{x=1}$.

(ii) If $f(x) = (\cos x)^{\sin x}$, calculate $\left.\dfrac{df}{dx}\right|_{x=0}$.

(iii) Calculate $\displaystyle\lim_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x}$.

Show your work in detail. Correct answers without detailed explanation do not get any credit.

Grading: 2+3+5=10 points.

Solution:

(i) First note that $x^x = \exp(x \ln x)$ and $(x^x)' = (x^x)(\ln x + 1)$.

Then we have $x^{x^x} = \exp(x^x \ln x)$ and

$$(x^{x^x})' = (x^{x^x})[(x^x)(\ln x + 1)\ln x + x^{x-1}].$$

Putting $x = 1$ we get $\left.\dfrac{df}{dx}\right|_{x=1} = 1$.

(ii) Since $(\cos x)^{\sin x} = \exp(\sin x \ln \cos x)$, we have

$$\big((\cos x)^{\sin x}\big)' = (\cos x)^{\sin x}[\cos x \ln \cos x - \frac{\sin^2 x}{\cos x}],$$

and putting $x = 0$ we get $\left.\dfrac{df}{dx}\right|_{x=0} = 0$.

(ii)

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x} &= \lim_{x\to\frac{\pi}{2}^-} \exp(\cos x \ln \tan x) = \exp(\lim_{x\to\frac{\pi}{2}^-} \cos x \ln \tan x) \\
&= \exp(\lim_{x\to\frac{\pi}{2}^-} \frac{\ln \tan x}{\sec x}) = \exp(\lim_{x\to\frac{\pi}{2}^-} \frac{\frac{\sec^2 x}{\tan x}}{\sec x \tan x}) \quad \textit{(Here L'Hospital's Rule is used)} \\
&= \exp(\lim_{x\to\frac{\pi}{2}^-} \frac{\cos x}{\sin^3 x}) \\
&= \exp(0) = 1.
\end{aligned}$$