Bilkent University

Quiz # 10
Math 101-Section 12 Calculus I
15 December 2022 Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Calculate the following:

(i) $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x-x\cos x}$.

(ii) $\displaystyle\int x^2 \ln x \, dx$.

(iii) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\, dx$.

Show your work in detail. Correct answers without detailed explanation do not get any credit.

Grading: 3+3+4=10 points.

Solution:

(i)

$$\lim_{x\to 0} \frac{x-\sin x}{x-x\cos x} \overset{LH}{=} \lim_{x\to 0} \frac{1-\cos x}{1-\cos x + x\sin x} \overset{LH}{=} \lim_{x\to 0} \frac{\sin x}{2\sin x + x\cos x} \overset{LH}{=} \lim_{x\to 0} \frac{\cos x}{3\cos x - x\sin x} = \frac{1}{3}.$$

(ii) Use integration by-parts with $u = \ln x$ and $dv = x^2$ to get

$$\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

(iii) First use integration by-parts with $u = \ln x$ and $dv = \dfrac{dx}{x^2}$ to get

$$\int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Since we have $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$ and $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$, we get

$$\int_1^\infty \frac{\ln x}{x^2}\, dx = \left(-\frac{\ln x}{x} - \frac{1}{x}\Big|_1^\infty \right) = 1.$$