**1.** Evaluate the following limit $\lim_{x\to 0}\left(\frac{1}{2x^2}+\frac{1}{x^3}+\frac{1}{x^4}-\frac{e^{\sin x}}{x^4}\right)$.

$$\lim_{x\to 0}\left(\frac{1}{2x^2}+\frac{1}{x^3}+\frac{1}{x^4}-\frac{e^{\sin x}}{x^4}\right)=\lim_{x\to 0}\frac{1+x+x^2/2-e^{\sin x}}{x^4}$$

$$\underset{\text{L'H}}{=}\lim_{x\to 0}\frac{1+x-e^{\sin x}\cdot\cos x}{4x^3}\underset{\text{L'H}}{=}\lim_{x\to 0}\frac{1-e^{\sin x}\cdot\cos^2 x+e^{\sin x}\cdot\sin x}{12x^2}$$

$$\underset{\text{L'H}}{=}\lim_{x\to 0}\frac{-e^{\sin x}\cdot\cos^3 x+e^{\sin x}\cdot 2\cos x\sin x+e^{\sin x}\cdot\cos x\sin x+e^{\sin x}\cos x}{24x}$$

$$=\lim_{x\to 0}\left(\frac{1}{8}\cdot\frac{\sin x}{x}\cdot\cos x\cdot e^{\sin x}+\frac{1}{24}\cdot\frac{1-\cos^2 x}{x}\cdot\cos x\cdot e^{\sin x}\right)$$

$$=\frac{1}{8}\cdot 1\cdot 1\cdot 1+\lim_{x\to 0}\left(\frac{1}{24}\cdot\frac{\sin x}{x}\cdot\sin x\cdot\cos x\cdot e^{\sin x}\right)$$

$$=\frac{1}{8}+\frac{1}{24}\cdot 1\cdot 0\cdot 1\cdot 1=\frac{1}{8}$$

**2a.** Evaluate the integral $\displaystyle\int \frac{dx}{(\tan x + \cot x)^2}$ .

$$\int \frac{dx}{(\tan x+\cot x)^2} = \int \frac{dx}{\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^2} = \int \frac{\sin^2 x \cos^2 x}{(\sin^2 x+\cos^2 x)^2}\, dx$$

$$= \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{4}\int \frac{1-\cos 4x}{2}\, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

**2b.** A function $f$ with a continuous derivative satisfies:

$$f(0) = 1\,, \qquad f(\pi/4) = 2\,, \qquad \int_0^{\pi/4} f(x)\,dx = 3\,, \qquad \int_0^{\pi/4} f(x)\tan^2 x\,dx = 4$$

Evaluate $\displaystyle\int_0^{\pi/4} f'(x)\tan x\,dx$ .

$$\int_0^{\pi/4} f'(x)\tan x\, dx = f(x)\tan x\Big]_0^{\pi/4} - \int_0^{\pi/4} f(x)\sec^2 x\, dx$$

$$\boxed{\begin{array}{l} u = \tan x \Rightarrow du = \sec^2 x\, dx \\ dv = f'(x)\,dx \Rightarrow v = f(x) \end{array}}$$

$$= f\left(\tfrac{\pi}{4}\right)\tan\left(\tfrac{\pi}{4}\right) - f(0)\tan(0) - \int_0^{\pi/4} f(x)(\tan^2 x+1)\, dx$$

$$= 2 - \int_0^{\pi/4} f(x)\tan^2 x\, dx - \int_0^{\pi/4} f(x)\, dx = 2-4-3 = -5$$

**3a.** The area of the region lying above the parabola $y = x^2$ and below a line passing through the origin is 1 square unit. Find all possible values of the slope of this line.

$$\left.\begin{matrix} y = mx \\ y = x^2 \end{matrix}\right\} \Rightarrow mx = x^2 \Rightarrow x = 0 \quad \text{or} \quad x = m$$

If $m > 0$, then: $\text{Area} = \int_0^m (mx - x^2)\,dx = \left[\frac{1}{2}mx^2 - \frac{1}{3}x^3\right]_0^m = \frac{1}{2}m^3 - \frac{1}{3}m^3 = \frac{1}{6}m^3$

So, $\text{Area} = 1 \Rightarrow m^3 = 6 \Rightarrow m = \sqrt[3]{6}$

If $m < 0$, then by symmetry $m = -\sqrt[3]{6}$.

**3b.** Find the volume of the solid generated by revolving the region between the graph of $y = 1/(1+e^x)$ and the $x$-axis for $x \geq 0$ about the $x$-axis.

$$\text{Volume} = \pi \int_0^\infty (\text{radius})^2\,dx = \pi \int_0^\infty \frac{1}{(1+e^x)^2}\,dx = \pi \int_0^\infty \frac{e^{-2x}}{(e^{-x}+1)^2}\,dx$$

$$= \pi \int_2^1 \frac{u-1}{u^2} \cdot (-du) = \pi \int_1^2 \left(\frac{1}{u} - \frac{1}{u^2}\right) du = \pi \left[\ln|u| + \frac{1}{u}\right]_1^2$$

$$\boxed{\begin{matrix} u = e^{-x} + 1 \\ du = -e^{-x}\,dx \end{matrix}}$$

$$= \pi \left(\ln 2 + \frac{1}{2} - \ln 1 - 1\right) = \pi \cdot \left(\ln 2 - \frac{1}{2}\right)$$

**4.** In each of the following, if the given statement is true for all $f$, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

**a.** If $f$ has a derivative on $(-\infty, \infty)$, then $f$ has an antiderivative on $(-\infty, \infty)$.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$

**b.** If $f$ has an antiderivative on $(-\infty, \infty)$, then $f$ has a derivative on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $|x|$

**c.** If $f'(x + 2\pi) = f'(x)$ for all $x$, then $f(x + 2\pi) = f(x)$ for all $x$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $x$

**d.** If $f(n) \geq n$ for all positive integers $n$, then $\lim_{x \to \infty} f(x) = \infty$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $x \cos(2\pi x)$

**e.** If $f$ is continuous on $(-\infty, \infty)$, then $\dfrac{d}{dx}\displaystyle\int_0^x f(xt)\,dt = f(x^2)$ for all $x$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $x$