1. Evaluate the following limits.

[Do not use L'Hôpital's Rule!]

a. $\lim_{x\to 1} \frac{2x^3-3x^2+1}{x^3-3x+2} = \lim_{x\to 1} \frac{(x-1)\cdot(2x^2-x-1)}{(x-1)\cdot(x^2+x-2)}$

$= \lim_{x\to 1} \frac{2x^2-x-1}{x^2+x-2} = \lim_{x\to 1} \frac{(x-1)\cdot(2x+1)}{(x-1)\cdot(x+2)} = \lim_{x\to 1} \frac{2x+1}{x+2} = \frac{2\cdot 1+1}{1+2} = 1$

b. $\lim_{x\to 0^-} \frac{\sqrt{\cos(x)-\cos(2x)}}{x}$

$\lim_{x\to 0} \frac{\cos x-\cos 2x}{x^2} = \lim_{x\to 0} \frac{\cos x-(2\cos^2 x-1)}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)\cdot(1+2\cos x)}{x^2}$

$= \lim_{x\to 0} \frac{2\sin^2(\frac{x}{2})}{x^2} \cdot \lim_{x\to 0}(1+2\cos x) = \frac{1}{2}\cdot\left(\lim_{x\to 0}\frac{\sin(\frac{x}{2})}{\frac{x}{2}}\right)^2 \cdot (1+2\cdot 1) = \frac{1}{2}\cdot 1^2\cdot 3 = \frac{3}{2}$

$\lim_{x\to 0^-} \frac{\sqrt{\cos x-\cos 2x}}{x} = \lim_{x\to 0^-}\left(-\sqrt{\frac{\cos x-\cos 2x}{x^2}}\right) = -\left(\lim_{x\to 0^-}\frac{\cos x-\cos 2x}{x^2}\right)^{1/2} = -\sqrt{\frac{3}{2}}$

2. Consider the function $f(x) = \dfrac{\sin(\pi x^3)}{x^2+1}$.

a. Find an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 1$.

$$f'(x) = \frac{\cos(\pi x^3)\cdot \pi \cdot 3x^2 \cdot (x^2+1) - \sin(\pi x^3)\cdot 2x}{(x^2+1)^2}$$

$$f'(1) = \frac{\cos(\pi)\cdot\pi\cdot 3\cdot 2 - \sin(\pi)\cdot 2}{2^2} = \frac{-1\cdot 6\pi - 0\cdot 2}{4} = -\frac{3\pi}{2}$$

$$f(1) = \frac{\sin(\pi)}{2} = 0$$

An equation for the tangent line is:

$$y - 0 = -\frac{3\pi}{2}\cdot(x-1)$$

b. Show that the equation $f(x) = \dfrac{3}{5}$ has at least two solutions.

$$f(0) = 0 < \frac{3}{5}$$

$$f(1) = 0 < \frac{3}{5}$$

$$f\left(\frac{1}{\sqrt[3]{2}}\right) = \frac{1}{2^{-2/3}+1} > \frac{3}{5}$$

because $32 > 27 \Rightarrow 2\cdot 2^{2/3} > 3 \Rightarrow \frac{2}{3} > 2^{-2/3} \Rightarrow \frac{5}{3} > 2^{-2/3}+1 \Rightarrow \frac{1}{2^{-2/3}+1} > \frac{3}{5}$.

f is continuous everywhere.

Hence, by IVT, there are c_1 in $\left(0, \frac{1}{\sqrt[3]{2}}\right)$ and c_2 in $\left(\frac{1}{\sqrt[3]{2}}, 1\right)$ such that $f(c_1) = 0$ and $f(c_2) = 0$.

3. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(0,1)}$ if y is a differentiable function of x satisfying the equation:

$$xy^2+\sin(\pi y)=x^3$$

$$\Big\Downarrow \; d/dx$$

$$y^2+2xy\frac{dy}{dx}+\pi\cos(\pi y)\frac{dy}{dx}=3x^2$$

$(x,y)=(0,1)$:

$$1+0\cdot\frac{dy}{dx}+\pi\cdot(-1)\frac{dy}{dx}=0$$

$$\Downarrow$$

$$\frac{dy}{dx}=\frac{1}{\pi} \text{ at } (x,y)=(0,1)$$

d/dx:

$$2y\frac{dy}{dx}+2y\frac{dy}{dx}+2x\left(\frac{dy}{dx}\right)^2+2xy\frac{d^2y}{dx^2}-\pi^2\sin(\pi y)\left(\frac{dy}{dx}\right)^2+\pi\cos(\pi y)\frac{d^2y}{dx^2}=6x$$

$$\Big\Downarrow \; (x,y)=(0,1),\ \frac{dy}{dx}=\frac{1}{\pi}$$

$$2\cdot\frac{1}{\pi}+2\cdot\frac{1}{\pi}+0\cdot\left(\frac{1}{\pi}\right)^2+0\cdot\frac{d^2y}{dx^2}-0\left(\frac{dy}{dx}\right)^2+\pi\cdot(-1)\frac{d^2y}{dx^2}=0$$

$$\Downarrow$$

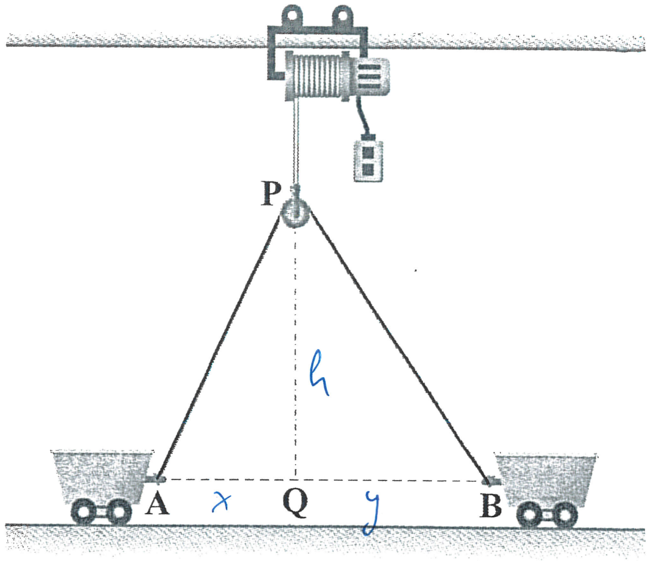
$$\frac{d^2y}{dx^2}=\frac{4}{\pi^2} \text{ at } (x,y)=(0,1)$$

4. A 13 m long rope is tied to two carts at the points A and B, and passes over a pulley at the point P as shown in the figure. A winch moves the pulley along a fixed vertical line. The point Q lies directly below the point P and on the line connecting the points A and B.

At a certain moment when the winch is raising the pulley at a rate of 1 m/s; P is 4 m away from Q, A is 3 m away from Q, and the distance between the points A and Q is decreasing at a rate of 2 m/s.

Determine how fast the distance between the points B and Q is changing at this moment.

[Do not assume anything about how the quantities in the question depend on time beyond what is already given in the question.]



At our moment: $x = 3\,m,\ \frac{dx}{dt} = -2\,m/s,\ h = 4\,m,\ \frac{dh}{dt} = 1\,m/s$

$$\sqrt{x^2+h^2} + \sqrt{y^2+h^2} = 13\,m \xRightarrow{\text{at our moment}} \sqrt{3^2+4^2} + \sqrt{y^2+4^2} = 13 \Rightarrow y = 4\sqrt{3}\,m$$

$$\Downarrow d/dt$$

$$\frac{2x\frac{dx}{dt} + 2h\frac{dh}{dt}}{2\sqrt{x^2+h^2}} + \frac{2y\frac{dy}{dt} + 2h\frac{dh}{dt}}{2\sqrt{y^2+h^2}} = 0$$

$$\Downarrow \leftarrow \text{at our moment}$$

$$\frac{3\cdot(-2)+4\cdot 1}{5} + \frac{4\sqrt{3}\cdot\frac{dy}{dt} + 4\cdot 1}{8} = 0$$

$$\Downarrow$$

$$\frac{dy}{dt} = -\frac{1}{5\sqrt{3}}\,m/s$$

The distance between B and Q is decreasing at a rate of $\frac{1}{5\sqrt{3}}$ m/s at this moment.