1. Sketch the graph of $y = x^{1/3} + x^{-2/3}$ by computing y' and y'', and determining their signs; finding the critical points, the inflections points, the intercepts, and the asymptotes; and clearly labeling them in the picture.

$$y' = \frac{1}{3}x^{-2/3} - \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x-2) = 0 \Rightarrow x = 2$$

undefined when $x = 0$.

$$y'' = -\frac{2}{9}x^{-5/3} + \frac{10}{9}x^{-8/3} = -\frac{2}{9}x^{-8/3}(x-5) = 0 \Rightarrow x = 5$$

x		0		2		5	
y'	+		−	0	+		+
y''	+		+		+	0	−

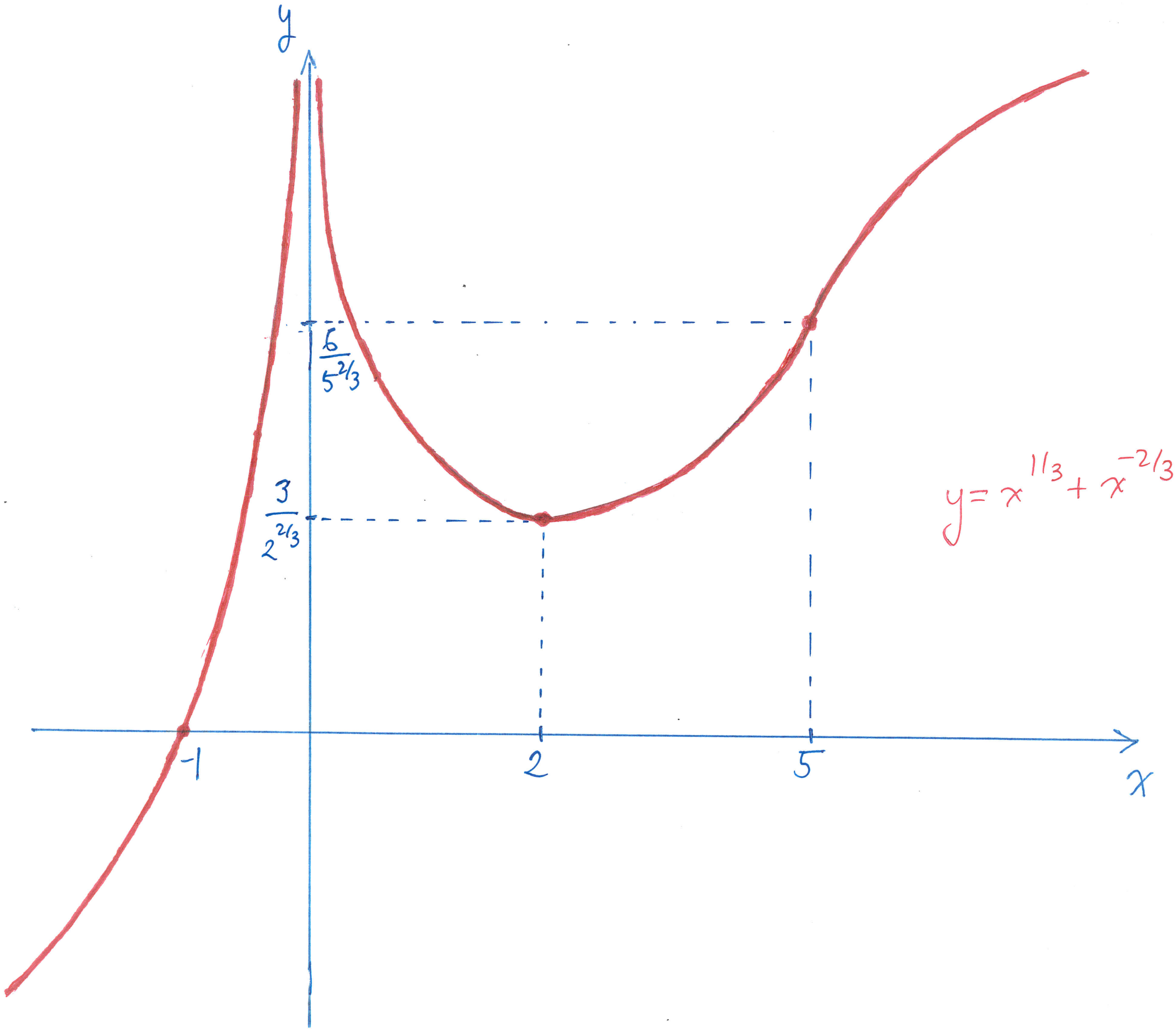
$\lim_{x \to 0} y = \infty$

$x = 2 \Rightarrow y = 3 \cdot 2^{-2/3}$

$x = 5 \Rightarrow y = 6 \cdot 5^{-2/3}$

$y = 0 \Rightarrow x^{1/3} + x^{-2/3} = 0$

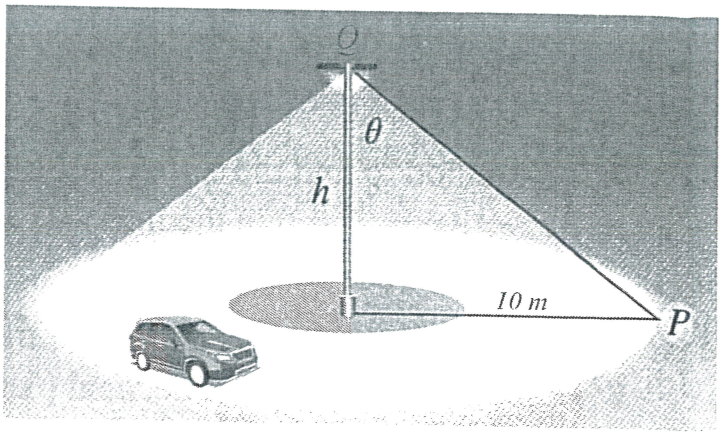
$\Rightarrow x^{-2/3}(x+1) = 0 \Rightarrow x = -1$



2. A lamp post will be erected in the center of a traffic circle with radius $10\,m$. The municipal regulations require the height h of the lamp post to be between $5\,m$ and $10\,m$. The intensity of illumination I at a point P at the edge of the traffic circle is given by

$$I = c\frac{\cos\theta}{(\text{the distance from } Q \text{ to } P)^2}$$

where c is a positive constant, and the point Q and the angle θ are as shown in the figure.



Determine how tall the lamp post should be to provide (a) the maximum and (b) the minimum illumination at P.

$|QP|^2 = h^2 + 10^2$ and $\cos\theta = \dfrac{h}{(h^2+10^2)^{1/2}}$.

Maximize/Minimize $I = c\cdot\dfrac{h}{(h^2+10^2)^{3/2}}$ for $5 \le h \le 10$

Critical points:

$$\frac{dI}{dh} = c\cdot\frac{1\cdot(h^2+10^2)^{3/2} - h\cdot\frac{3}{2}\cdot(h^2+10^2)^{1/2}\cdot 2h}{(h^2+10^2)^3} = c\cdot\frac{10^2-2h^2}{(h^2+10^2)^{5/2}}$$

$$\frac{dI}{dh} = 0 \Rightarrow 10^2 - 2h^2 = 0 \Rightarrow h = 5\sqrt{2} \text{ or } \underbrace{h = -5\sqrt{2}}_{\text{not in the interval}} \otimes$$

$$\Downarrow$$

$$I = \frac{c}{150\sqrt{3}}$$

Endpoints:

$$h = 5 \Rightarrow I = \frac{c}{125\sqrt{5}}$$

$$h = 10 \Rightarrow I = \frac{c}{200\sqrt{2}}$$

$$128 > 125 > 108 \Rightarrow 8\sqrt{2} > 5\sqrt{5} > 6\sqrt{3} \Rightarrow \frac{1}{200\sqrt{2}} < \frac{1}{125\sqrt{5}} < \frac{1}{150\sqrt{3}}$$

$\Rightarrow \begin{cases} \text{A } 5\sqrt{2}\,m \text{ tall lamp provides the maximum illumination.} \\ \text{A } 10\,m \text{ tall lamp provides the minimum illumination.} \end{cases}$

3. Evaluate the following integrals.

a. $\displaystyle\int \frac{dx}{\tan x + \cot x} = \int \frac{1}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}\, dx = \int \frac{\sin x \cos x}{\sin^2 x + \cos^2 x}\, dx$

$\displaystyle = \frac{1}{2} \int \sin 2x \, dx = -\frac{1}{4} \cos 2x + C$

b. $\displaystyle\int_{1/3}^{1} x^{-4}(x^{-7} + 7x^{-3})^{1/3}\, dx = \int_{1/3}^{1} (x^{-4} + 7)^{1/3} \cdot x^{-5}\, dx$

$\displaystyle = \int_{88}^{8} u^{1/3} \cdot \left(-\frac{1}{4}\, du\right) = -\frac{1}{4} \cdot \frac{u^{4/3}}{4/3}\Bigg]_{88}^{8} = -\frac{3}{16} \cdot \left(8^{4/3} - 88^{4/3}\right)$

$\boxed{\begin{array}{l} u = x^{-4} + 7 \\ du = -4x^{-5}\, dx \end{array}}$

$= 3 \cdot (11^{4/3} - 1)$

4a. Find $y(4)$ if $\frac{dy}{dx} = \sqrt{xy}$ and $y(0) = \frac{1}{9}$.

$$\Downarrow$$

$$\frac{dy}{\sqrt{y}} = \sqrt{x}\,dx \overset{\int}{\Longrightarrow} \underbrace{2\sqrt{y} = \frac{x^{3/2}}{3/2} + C' \xRightarrow{\boxed{x=0,\ y=\frac{1}{9}}} \frac{2}{3} = C'}$$

$$\Downarrow$$

$$y = \left(\frac{x^{3/2}+1}{3}\right)^2$$

$$\Downarrow$$

$$y(4) = \left(\frac{4^{3/2}+1}{3}\right)^2 = 9$$

4b. Suppose f is a continuous function such that

- $f(0) = 2017$, $f(2) = 2020$, and $f(4) = 2023$, and
- the average value of f on the interval $[0,4]$ is 1.

Find $g'(2)$ where $g(x) = \int_0^x f(xt)\,dt$.

$$g(x) = \int_0^x f(xt)\,dt \underset{\boxed{\substack{u=xt\\ du=x\,dt}}}{=} \int_0^{x^2} f(u)\cdot\left(\frac{1}{x}\,du\right) = \frac{1}{x}\int_0^{x^2} f(u)\,du$$

$$g'(x) = \frac{d}{dx}\left(\frac{1}{x}\int_0^{x^2} f(u)\,du\right) = -\frac{1}{x^2}\int_0^{x^2} f(u)\,du + \frac{1}{x}\cdot\frac{d}{dx}\left(\int_0^{x^2} f(u)\,du\right)$$

$$= -\frac{1}{x^2}\int_0^{x^2} f(u)\,du + \frac{1}{x}\cdot f(x^2)\cdot\frac{d(x^2)}{dx} = -\frac{1}{x^2}\int_0^{x^2} f(u)\,du + 2f(x^2)$$

(Leibniz Rule)

$$g'(2) = -\underbrace{\frac{\int_0^4 f(u)\,du}{4}}_{\text{average of } f \text{ on } [0,4]} + 2f(4) = -1 + 2\cdot 2023 = 4045$$