Bilkent University

Quiz # 07
Math 101-Section 05 Calculus I
9 November 2023 Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Let $f(x) = x^4$ on the interval $[0,1]$. Subdivide this interval into n equal subintervals as $0 = x_0 < x_1 < \cdots < x_n = 1$. For this function and for this partition we define $L_n = \sum_{i=0}^{n-1} \frac{1}{n} f(x_i)$, $U_n = \sum_{i=1}^{n} \frac{1}{n} f(x_i)$, $R_n = \sum_{i=1}^{n} \frac{1}{n} f(x_i^*)$, where each $x_i^* \in [x_{i-1}, x_i]$ is an arbitrarily chosen points.

(a) Calculate $\lim_{n\to\infty} L_n$. **(b)** Calculate $\lim_{n\to\infty} U_n$. **(c)** Calculate $\lim_{n\to\infty} R_n$.

Hint: $1^4 + \cdots + n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$.

Grading: 3+3+4=10 points

Solution: (Grader: `taha.yigit@ug.bilkent.edu.tr`)

(a) Note that $x_i = i/n$. Then we have

$$L_n = \sum_{i=0}^{n-1} \frac{1}{n}\frac{i^4}{n^4} = \frac{1}{n^5}\sum_{i=0}^{n-1} i^4 = \frac{1}{n^5}[\frac{1}{5}(n-1)^5 + \frac{1}{2}(n-1)^4 + \frac{1}{3}(n-1)^3 - \frac{1}{30}(n-1)]$$

Now we clearly have $\lim_{n\to\infty} L_n = \frac{1}{5}$.

(b) As above we have $x_i = i/n$ and we have

$$U_n = \sum_{i=1}^{n} \frac{1}{n}\frac{i^4}{n^4} = \frac{1}{n^5}\sum_{i=1}^{n} i^4 = \frac{1}{n^5}[\frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n]$$

Now we have $\lim_{n\to\infty} U_n = \frac{1}{5}$.

(c) Since f is increasing we have $L_n < R_n < U_n$.
Using the squeeze theorem we have $\lim_{n\to\infty} R_n = \frac{1}{5}$.