Bilkent University

Q-1) Let $f(x) = \dfrac{x}{\ln x}$, for $1 < x < \infty$.

   (a) Find the minimum value of $f(x)$ on $(1, \infty)$.

   (b) Let $a > 1$, $a \neq e$, be a real number. Find out which one is larger, e^a or a^e.

   (c) Find out which one is larger, e^ϕ or ϕ^e, where $\phi = \dfrac{1 + \sqrt{5}}{2} \approx 1.6$ is the Golden ratio.

   Show your work in detail. Correct answers with no justification will not get any credit.

   Grading: 4+5+1=10 points

Solution: (Grader: `rburakguler71@gmail.com`)

(a) $f'(x) = \dfrac{\ln x - 1}{(\ln x)^2} = 0$ when $x = e$.

$f''(x) = \dfrac{2 - \ln x}{x(\ln x)^3}$, and $f''(e) = 1/e > 0$. Hence by the second derivative test $x = e$ is a local minimum point. Being the only critical point, it is the global minimum point.

You can also conclude that the only critical point is the global minimum point after observing that $\lim\limits_{x \to 1^+} f(x) = \infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$.

Hence the minimum value of $f(x)$ on $(1, \infty)$ is $f(e) = e$.

(b) We now know from part (a) that $e < \dfrac{x}{\ln x}$ for any $x > 1$ but $x \neq e$. In particular for any $a > 1$ and $a \neq e$, we have

$$e < \frac{a}{\ln a}$$
$$e \ln a < a$$
$$\ln a^e < a$$
$$a^e < e^a.$$

Hence e^a is the larger one.

(c) From part (b) we immediately see that $\phi^e < e^\phi$.