1. Let R be the region between the curves $y = 2\cos(x/2)$ and $y = \sin(x)$ for $0 \leq x \leq \pi$.

- Let V be the volume obtained by rotating R about the x-axis.
- Let W be the volume obtained by rotating R about the y-axis.

a. Express V as an integral using the washer method.

$$V = \pi \int_0^{\pi} \left(\left(2\cos\left(\frac{x}{2}\right)\right)^2 - (\sin(x))^2 \right) dx$$

b. Express W as an integral using the cylindrical shells method.

$$W = 2\pi \int_0^{\pi} x \cdot \left(2\cos\left(\frac{x}{2}\right) - \sin(x) \right) dx$$

c. Compute either ☒ V or ☐ W.

[Indicate the one you choose to compute by putting a ✗ in the ☐ to the left of it. Choose only one. You will get points only for the one you choose. If you mark both or neither, then this part won't be graded.]

$$V = \pi \int_0^{\pi} \left(4\cos^2\left(\frac{x}{2}\right) - \sin^2 x \right) dx = \pi \int_0^{\pi} \left(4 \cdot \frac{1+\cos x}{2} - \frac{1-\cos 2x}{2} \right) dx$$

$$= \pi \int_0^{\pi} \left(\frac{3}{2} + 2\cos x + \frac{1}{2}\cos 2x \right) dx$$

$$= \pi \cdot \left[\frac{3}{2}x + 2\sin x + \frac{1}{4}\sin 2x \right]_0^{\pi}$$

$$= \pi \cdot \left\{ \left(\frac{3}{2}\pi + 2\sin\pi + \frac{1}{4}\sin 2\pi \right) - \left(0 + 2\sin 0 + \frac{1}{4}\sin 0 \right) \right\}$$

$$= \pi \cdot \left(\frac{3}{2}\pi - 0 \right) = \frac{3}{2}\pi^2$$

2a. Evaluate the limit $\lim_{x\to 0} \frac{1+\ln(1+x)-\sqrt{2}\sin(x+\pi/4)}{x^3}$.

$$\lim_{x\to 0} \frac{1+\ln(1+x)-\sqrt{2}\sin(x+\pi/4)}{x^3} \overset{\text{l'H}}{\underset{\downarrow}{=}} \lim_{x\to 0} \frac{\frac{1}{1+x}-\sqrt{2}\cos(x+\pi/4)}{3x^2}$$

$$\overset{\text{l'H}}{\underset{\downarrow}{=}} \lim_{x\to 0} \frac{-\frac{1}{(1+x)^2}+\sqrt{2}\sin(x+\pi/4)}{6x} \overset{\text{l'H}}{\underset{\downarrow}{=}} \lim_{x\to 0} \frac{\frac{2}{(1+x)^3}+\sqrt{2}\cos(x+\pi/4)}{6}$$

$$= \frac{2+\sqrt{2}\cdot\frac{1}{\sqrt{2}}}{6} = \frac{3}{6} = \frac{1}{2}$$

2b. Evaluate the integral $\int_0^\infty \frac{dx}{(x^2+1)(3x^2+1)}$.

$$\int \frac{dx}{(x^2+1)(3x^2+1)} = \int \frac{1}{2}\left(\frac{3}{3x^2+1}-\frac{1}{x^2+1}\right)dx = \frac{\sqrt{3}}{2}\arctan(\sqrt{3}x)-\frac{1}{2}\arctan(x)+C$$

$$\int_0^\infty \frac{dx}{(x^2+1)(3x^2+1)} = \lim_{c\to\infty}\int_0^c \frac{dx}{(x^2+1)(3x^2+1)} = \lim_{c\to\infty}\left[\frac{\sqrt{3}}{2}\arctan(\sqrt{3}x)-\frac{1}{2}\arctan(x)\right]_0^c$$

$$= \lim_{c\to\infty}\left(\frac{\sqrt{3}}{2}\arctan(\sqrt{3}c)-\frac{1}{2}\arctan(c)-\frac{\sqrt{3}}{2}\arctan(0)+\frac{1}{2}\arctan(0)\right)$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\frac{\pi}{2} = (\sqrt{3}-1)\cdot\frac{\pi}{4}$$

3. Consider the function $f(x) = \exp\big((\ln x)^3\big) = e^{\left((\ln x)^3\right)}$.

a. Compute $f'(x)$.

$$f'(x) = e^{(\ln x)^3} \cdot 3(\ln x)^2 \cdot \frac{1}{x}$$

b. Compute $f''(x)$.

$$f''(x) = e^{(\ln x)^3} \cdot 9(\ln x)^4 \cdot \frac{1}{x^2} + e^{(\ln x)^3} \cdot 3 \cdot \left(2\ln x \cdot \frac{1}{x^2} + (\ln x)^2 \cdot \left(-\frac{1}{x^2}\right)\right)$$

c. Sketch the graph of $y = f(x)$.

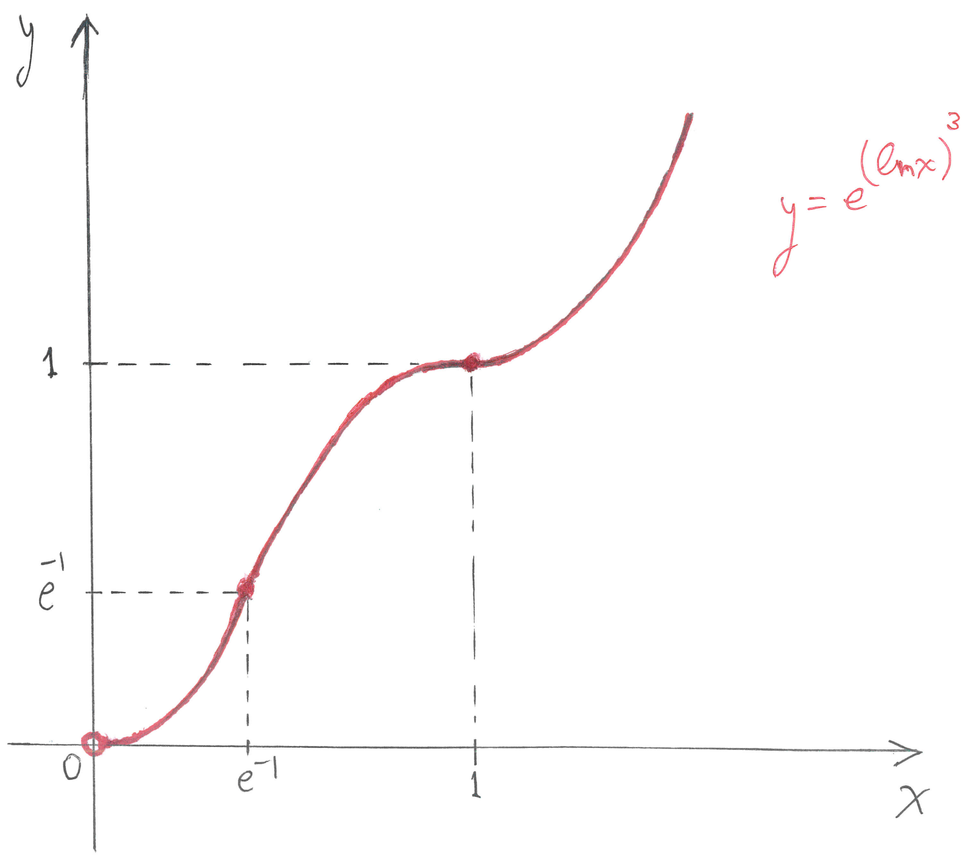
$$f'(x) = 0 \Rightarrow x = 1$$

$$f''(x) = e^{(\ln x)^3} \cdot 3 \cdot \frac{1}{x^2} \cdot \ln x \cdot \underbrace{\left(3(\ln x)^3 - \ln x + 2\right)}_{(\ln x + 1)(3(\ln x)^2 - 3\ln x + 2)} = 0 \Rightarrow x = 1 \text{ or } x = e^{-1}$$

x	0		e^{-1}		1	
f'		+		+	0	+
f''		+	0	−	0	+

$f(1) = 1$, $f(e^{-1}) = e^{-1}$

$\lim_{x \to 0^+} f(x) = 0$



4. In each of the following, if the given statement is true for all f, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

a. If f is one-to-one and differentiable on $(-\infty, \infty)$, then f^{-1} is differentiable.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ x^3

b. If f is differentiable on $(-\infty, \infty)$, then f has an antiderivative on $(-\infty, \infty)$.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$

c. If f has an antiderivative on $(-\infty, \infty)$, then f is differentiable on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $x^{1/3}$.

d. If $\int_0^\infty f(x)\,dx = 1$, then $g(x) = \int_0^\infty f(t/x)\,dt$ is increasing on $(0, \infty)$.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$

e. If f is continuous and $f(x + 2\pi) = f(x)$ for all x, then $\int_0^{x+2\pi} f(t)\,dt = \int_0^x f(t)\,dt$ for all x.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ 1