1. Evaluate the following limits. [Do not use L'Hôpital's Rule!]

a. $\lim_{x\to 0^+} \frac{(x^{1/2}-x)^{1/2}-x^{1/4}}{x^{3/4}} = \lim_{x\to 0^+} \frac{x^{1/4}\cdot(1-x^{1/2})^{1/2}-x^{1/4}}{x^{3/4}}$

$= \lim_{x\to 0^+} \frac{(1-x^{1/2})^{1/2}-1}{x^{1/2}} = \lim_{x\to 0^+} \frac{((1-x^{1/2})^{1/2}-1)\cdot((1-x^{1/2})^{1/2}+1)}{x^{1/2}\cdot((1-x^{1/2})^{1/2}+1)}$

$= \lim_{x\to 0^+} \frac{1-x^{1/2}-1}{x^{1/2}\cdot((1-x^{1/2})^{1/2}+1)} = -\lim_{x\to 0^+} \frac{x^{1/2}}{x^{1/2}\cdot((1-x^{1/2})^{1/2}+1)}$

$= -\lim_{x\to 0^+} \frac{1}{(1-x^{1/2})^{1/2}+1} = -\frac{1}{1+1} = -\frac{1}{2}$

b. $\lim_{x\to 0} \frac{\sin(x^2)\cos(\cot x)}{x}$

$\lim_{x\to 0} \frac{\sin(x^2)}{x} = \lim_{x\to 0}\left(\frac{\sin(x^2)}{x^2}\cdot x\right) = \lim_{x\to 0}\frac{\sin(x^2)}{x^2}\cdot \lim_{x\to 0} x = 1\cdot 0 = 0$

$|\cos(\cot x)| \leq 1$ for $0<|x|<\pi$

$\Downarrow$

$\left|\frac{\sin(x^2)}{x}\cdot\cos(\cot x)\right| \leq \left|\frac{\sin(x^2)}{x}\right|$ for $0<|x|<\pi$

$\Downarrow$

$-\left|\frac{\sin(x^2)}{x}\right| \leq \frac{\sin(x^2)\cos(\cot x)}{x} \leq \left|\frac{\sin(x^2)}{x}\right|$ for $0<|x|<\pi$

Since $\lim_{x\to 0}\left|\frac{\sin(x^2)}{x}\right| = 0 = \lim_{x\to 0}\left(-\left|\frac{\sin(x^2)}{x}\right|\right)$, by Squeeze Theorem

$\lim_{x\to 0}\frac{\sin(x^2)\cos(\cot x)}{x} = 0$

2. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(0,1)}$ if y is a differentiable function of x satisfying the equation $y^3 = y^2\cos x + \sin x$.

$$y^3 = y^2 \cos x + \sin x$$

$$\Big\Downarrow \; d/dx$$

$$3y^2 \cdot y' = 2y \cdot y' \cdot \cos x - y^2 \sin x + \cos x$$

$$\xrightarrow{(x,y)=(0,1)}$$

$$3y' = 2y' - 0 + 1 \Rightarrow y' = 1 \text{ at } (x,y) = (0,1)$$

$$\Big\Downarrow \; d/dx$$

$$6y \cdot (y')^2 + 3y^2 \cdot y'' = 2(y')^2 \cos x + 2y \cdot y'' \cos x - 2y \cdot y' \sin x - 2y \cdot y' \sin x - y^2 \cos x - \sin x$$

$$\Big\Downarrow \; (x,y) = (0,1),\ y' = 1$$

$$6 + 3y'' = 2 + 2y'' - 0 - 0 - 1 - 0$$

$$\Big\Downarrow$$

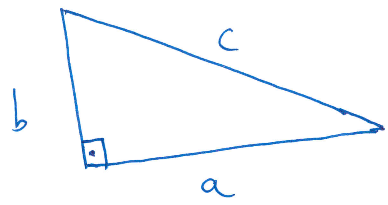
$$y'' = -5 \text{ at } (x,y) = (0,1)$$

3. The lengths of the sides of a right triangle are changing as differentiable functions of time. At a certain moment,

- the length of one of the legs is increasing at a rate of 1 cm/s,
- the length of the other leg is decreasing at a rate of 2 cm/s,
- the length of the hypotenuse is decreasing at a rate of 1 cm/s, and
- the area of the triangle is decreasing at a rate of 1/2 cm^2/s.

Find the length of the shortest side at this moment.

Let a, b, c be the lengths of the sides and let A be the area of the triangle.



Then:

$$\frac{da}{dt} = 1 \text{ cm/s}, \quad \frac{db}{dt} = -2 \text{ cm/s}, \quad \frac{dc}{dt} = -1 \text{ cm/s}, \quad \frac{dA}{dt} = -\frac{1}{2} \text{ cm}^2/\text{s}$$

at our moment.

$$a^2 + b^2 = c^2 \xRightarrow{d/dt} 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$\Downarrow$$

$$a - 2b = -\sqrt{a^2+b^2} \qquad (1)$$

$$A = \frac{1}{2}ab \xRightarrow{d/dt} \frac{1}{2}a\frac{db}{dt} + \frac{1}{2}b\frac{da}{dt} = \frac{dA}{dt}$$

$$\Downarrow$$

$$-2a + b = -1 \qquad (2)$$

$(1) \Rightarrow \quad a^2 - 4ab + 4b^2 = a^2 + b^2 \Rightarrow 3b^2 = 4ab \Rightarrow b = 0$ or $b = \frac{4}{3}a$

$b = 0 \xRightarrow{(2)} a = \frac{1}{2}$, but $(a,b) = (\frac{1}{2}, 0)$ is not a solution of (1).

$b = \frac{4}{3}a \xRightarrow{(2)} a = \frac{3}{2} \xRightarrow{(2)} b = 2$

The shortest side is $\frac{3}{2}$ cm long at this moment.

4. Let $f(x) = 2x^4 - 7x^2 + 6x$.

a. Compute $f'(x)$.

$f'(x) = 8x^3 - 14x + 6$

b. Write an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = -1$.

☞ *Write only the equation (and nothing else) in the box!*

$$\boxed{y-(-11)=12\cdot(x-(-1))}$$

$f'(-1)=12$, $f(-1)=-11$

c. Find the absolute maximum and minimum values of f on the interval $[-2, 2]$.

$f'(x) = 2\cdot(4x^3-7x+3) = 2\cdot(x-1)\cdot(4x^2+4x-3) = 2\cdot(x-1)(2x-1)(2x+3)$

$f'(x)=0 \Rightarrow x=1$ or $x=\frac{1}{2}$ or $x=-\frac{3}{2}$

Critical points

$x=1 \Rightarrow f(1)=1$

$x=\frac{1}{2} \Rightarrow f(\frac{1}{2})=\frac{11}{8}$

$x=-\frac{3}{2} \Rightarrow f(-\frac{3}{2})=-\frac{117}{8}$

Endpoints

$x=-2 \Rightarrow f(-2)=-8$

$x=2 \Rightarrow f(2)=16$

The absolute maximum value is 16.

The absolute minimum value is $-\frac{117}{8}$.