1. Sketch the graph of $y = 1 - 6x^{-2} + 4x^{-3}$ by computing y' and y'', and determining their signs and the corresponding shapes; finding the critical points, the inflections points, the intercepts, and the asymptotes; and clearly labeling them in the picture.

$$y' = 12x^{-3} - 12x^{-4} = 12x^{-4} \cdot (x-1) = 0 \Rightarrow x = 1 \Rightarrow y = -1$$

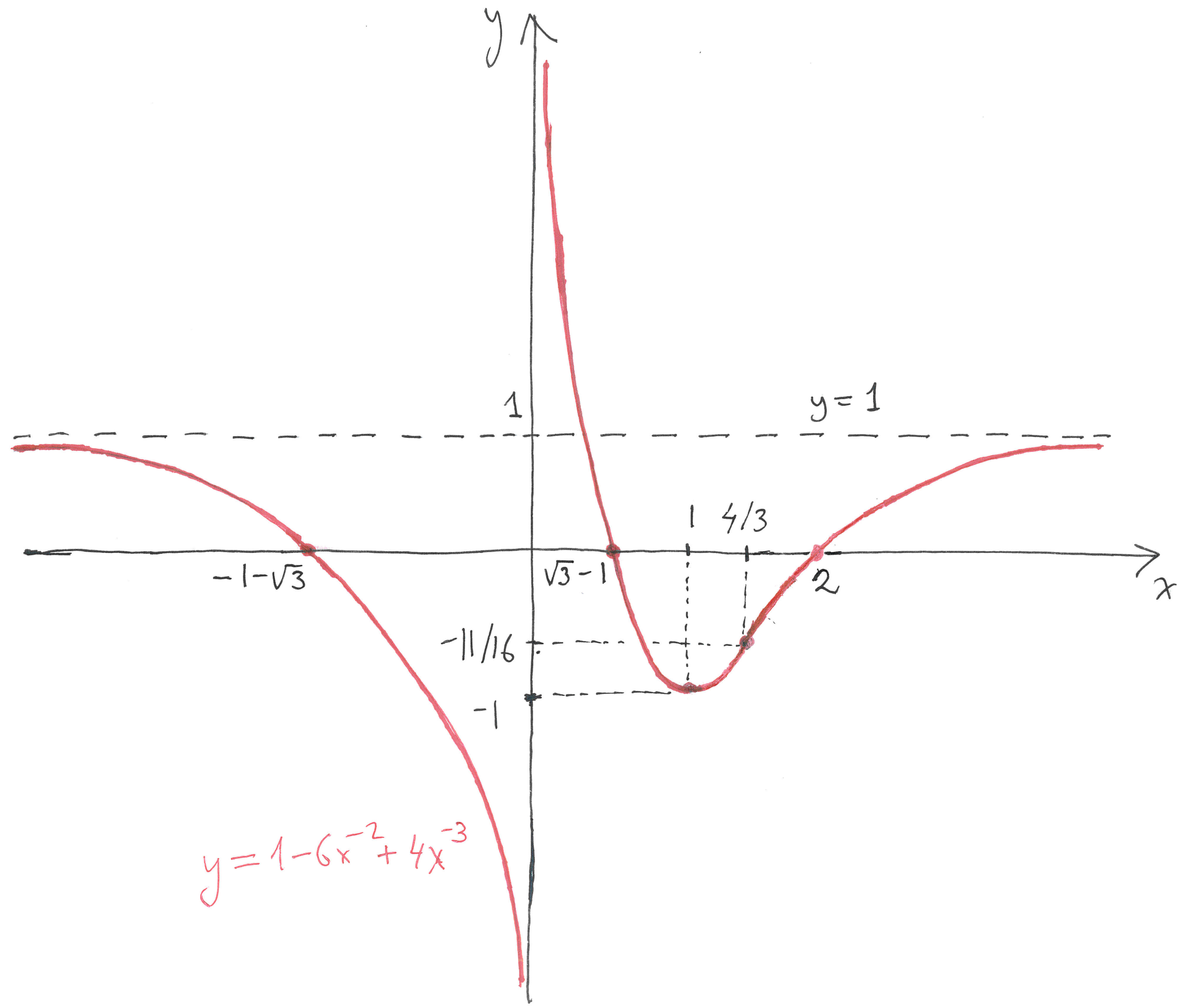
$$y'' = -36x^{-4} + 48x^{-5} = -36x^{-5} \cdot \left(x - \frac{4}{3}\right) = 0 \Rightarrow x = \frac{4}{3} \Rightarrow y = -\frac{11}{16}$$

$$y = 0 \Rightarrow 1 - 6x^{-2} + 4x^{-3} = 0 \Rightarrow x^3 - 6x + 4 = 0 \Rightarrow (x-2)(x^2+2x-2) = 0 \Rightarrow x = 2, \; x = -1 \pm \sqrt{3}$$

x		0		1		4/3	
y'	$-$		$-$	0	$+$		$+$
y''	$-$		$+$		$+$	0	$-$

$$\lim_{x\to\infty} y = 1, \quad \lim_{x\to-\infty} y = 1$$

$$\lim_{x\to0^+} y = \infty, \quad \lim_{x\to0^-} y = -\infty$$



2. We want to design a flower bed in the shape of a half-disk with diameter $|AB| = 20$ m. This half-disk will be separated into three regions by the line segments $[AC]$ and $[BC]$, and roses, hydrangeas and peonies will be planted in these regions, as shown in the figure.

The costs of planting roses, hydrangeas and peonies are 20 ₺/m², 30 ₺/m², and 50 ₺/m², respectively. Find the costs of the most expensive and the least expensive flower beds we can design.

$$\text{Cost} = 20\cdot\left(\tfrac{1}{2}\cdot 20\cdot 10\sin\theta\right) + 30\cdot\left(\tfrac{1}{2}\cdot 10^2\cdot(\pi-\theta) - \tfrac{1}{2}\cdot 10\cdot 10\sin\theta\right)$$
$$+50\cdot\left(\tfrac{1}{2}\cdot 10^2\cdot\theta - \tfrac{1}{2}\cdot 10\cdot 10\sin\theta\right)$$

Maximize/Minimize $\text{Cost} = \left(\frac{3\pi}{2} + \theta - 2\sin\theta\right)\times 10^3$ for $0 \le \theta \le \pi$

Critical points: $\frac{d}{d\theta}\text{Cost} = 1 - 2\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{3} \Rightarrow \text{Cost} = \left(\frac{11\pi}{6} - \sqrt{3}\right)\times 10^3$ ₺

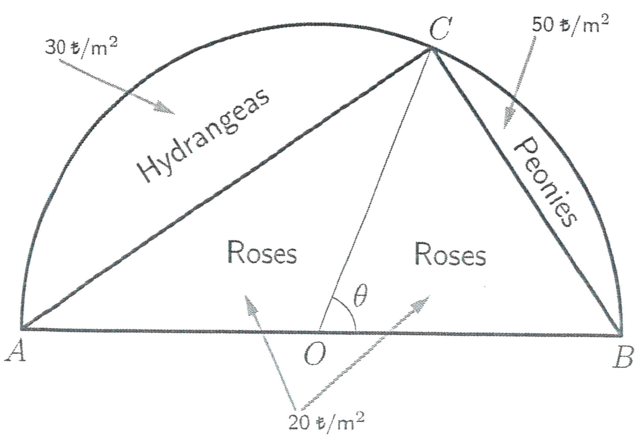
Endpoints: $\theta = 0 \Rightarrow \text{Cost} = \frac{3\pi}{2}\times 10^3$ ₺

$\theta = \pi \Rightarrow \text{Cost} = \frac{5\pi}{2}\times 10^3$ ₺

Since $3\sqrt{3} > 5 > \pi$, we have $\frac{5\pi}{2} > \frac{3\pi}{2} > \frac{11\pi}{6} - \sqrt{3}$.

The most expensive flower bed costs $\frac{5\pi}{2}\times 10^3$ ₺

The least expensive flower bed costs $\left(\frac{11\pi}{6} - \sqrt{3}\right)\times 10^3$ ₺



3a. Find the area of the region between the curves $y = 3\sec^2 x$ and $y = 8\cos 2x$ for $0 \le x \le \pi/4$.

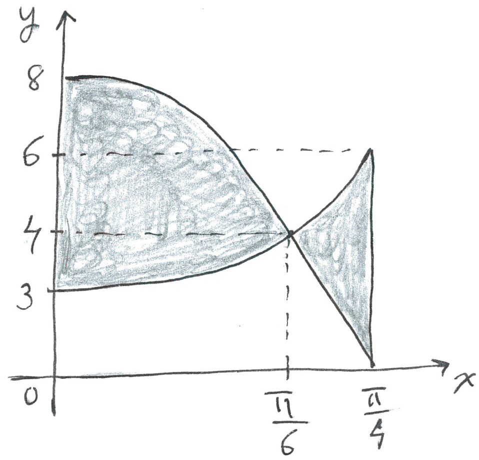
$$\text{Area} = \int_0^{\pi/6} (8\cos 2x - 3\sec^2 x)\,dx + \int_{\pi/6}^{\pi/4} (3\sec^2 x - 8\cos 2x)\,dx$$

$$= \Big[4\sin 2x - 3\tan x\Big]_0^{\pi/6} + \Big[3\tan x - 4\sin 2x\Big]_{\pi/6}^{\pi/4}$$

$$= 4\sin\frac{\pi}{3} - 3\tan\frac{\pi}{6} + 3\tan\frac{\pi}{4} - 4\sin\frac{\pi}{2} - 3\tan\frac{\pi}{6} + 4\sin\frac{\pi}{3}$$

$$= 4\cdot\frac{\sqrt{3}}{2} - 3\cdot\frac{1}{\sqrt{3}} + 3\cdot 1 - 4\cdot 1 - 3\cdot\frac{1}{\sqrt{3}} + 4\cdot\frac{\sqrt{3}}{2}$$

$$= 2\sqrt{3} - 1$$



3b. Evaluate the integral $\displaystyle\int \left(x^3 + 1 - \frac{2}{x^3}\right)\sqrt{x^2 + \frac{2}{x}}\,dx.$

$$\int \left(x^3 + 1 - \frac{2}{x^3}\right)\sqrt{x^2 + \frac{2}{x}}\,dx = \int \left(x - \frac{1}{x^2}\right)\left(x^2 + \frac{2}{x}\right)\sqrt{x^2 + \frac{2}{x}}\,dx$$

$$= \int u^{3/2}\cdot\frac{1}{2}\,du = \frac{1}{2}\cdot\frac{u^{5/2}}{5/2} + C' = \frac{1}{5}\cdot\left(x^2 + \frac{2}{x}\right)^{5/2} + C$$

$$\boxed{\begin{aligned} u &= x^2 + \frac{2}{x} \\ du &= \left(2x - \frac{2}{x^2}\right)dx \end{aligned}}$$

4a. Suppose that f is differentiable on $(-\infty, \infty)$ and the slope of the tangent line at every point (x, y) on the graph of $y = f(x)$ is equal to x/y. Find $f(2)$ if $f(1) = 2$.

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C'$$

$$\xrightarrow[(x,y)=(1,2)]{} \frac{1}{2}\cdot 2^2 = \frac{1}{2}\cdot 1^2 + C' \Rightarrow C' = \frac{3}{2}$$

Hence $f(x)^2 = x^2 + 3$ for all x.

f is differentiable on $(-\infty, \infty) \Rightarrow f$ is continuous on $(-\infty, \infty)$

$f(x)^2 = x^2 + 3 \geq 3$ for all $x \Rightarrow f(x) \neq 0$ for all x

$\Rightarrow$ By IVT, $f(x) > 0$ for all x or $f(x) < 0$ for all x

Since $f(1) = 2 > 0$, we have $f(x) > 0$ for all x

Hence $f(x) = \sqrt{x^2+3}$ for all x. Therefore $f(2) = \sqrt{7}$.

4b. Suppose that a continuous function f satisfies

$$\int_0^x f(t)\,dt + \int_0^{\sqrt{x}} f(t^2)\,dt + \int_0^{\sqrt[3]{x}} f(t^3)\,dt = x$$

for all $x \geq 0$. Find $f(64)$.

$$\Downarrow d/dx$$

$$f(x) + f((\sqrt{x})^2)\,\frac{d\sqrt{x}}{dx} + f((\sqrt[3]{x})^3)\cdot\frac{d\sqrt[3]{x}}{dx} = 1 \quad \text{by Leibniz Rule}$$

$$\Downarrow$$

$$f(x)\cdot\left(1 + \frac{1}{2}x^{-1/2} + \frac{1}{3}x^{-2/3}\right) = 1$$

$$\Downarrow x = 64$$

$$f(64)\cdot\left(1 + \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{3}\cdot\frac{1}{16}\right) = 1$$

$$\Downarrow$$

$$f(64) = \frac{12}{13}$$