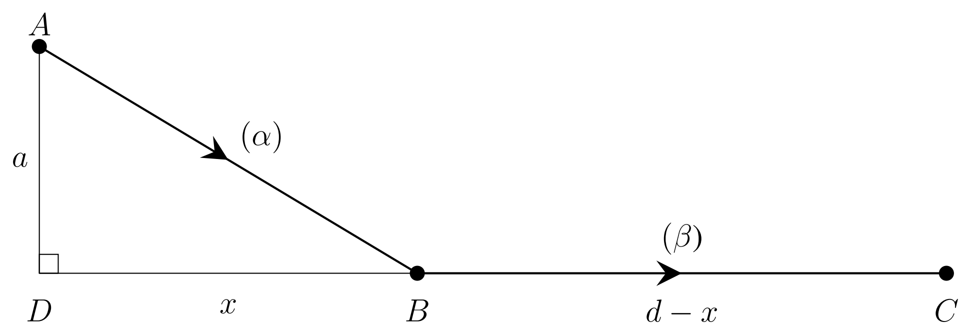
Quiz # 05
Math 101 Section 03 Calculus I
13 November 2024 Wednesday
Instructor: Ali Sinan Sertöz
Solution Key

Bilkent University

Q-1) The cost of a pipeline from A to B is α TL/km, and from B to C is β TL/km. Here $0 < a, d$ are in km units, $0 < \alpha, \beta$ are real numbers, and DBC is a straight line. Let $f(x)$ denote the function that gives the total cost of a pipeline from A to C via B as in the figure below.



(i) Write $f(x)$ explicitly together with its domain.
(ii) Calculate $f'(x)$.
(iii) Find x_0 such that $f'(x_0) = 0$.
(iv) Calculate $f(0)$, and $f(d)$.
(v) Calculate the minimum value of f when $(a, d, \alpha, \beta) = (7, 24, 17, 15)$.
(vi) Calculate the minimum value of f when $(a, d, \alpha, \beta) = (7, 24, 15, 17)$.

Hint: $7^2 + 24^2 = 25^2$, $8^2 + 15^2 = 17^2$ and $13 < \frac{105}{8} < 14$.

Grading: 1+1+2+2+2+2=10 points

Solution: Grader: gunes.akbas@bilkent.edu.tr

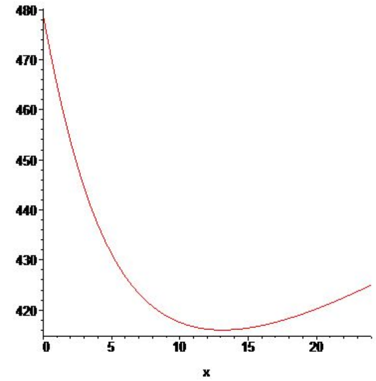
(i) $f(x) = \alpha\sqrt{x^2 + a^2} + \beta(d - x)$, $x \in [0, d]$.

(ii) $f'(x) = \dfrac{\alpha x}{\sqrt{x^2 + a^2}} - \beta$ **(iii)** $x_0 = \dfrac{\beta a}{\sqrt{\alpha^2 - \beta^2}}$ when $\alpha > \beta$, and does not exist otherwise.

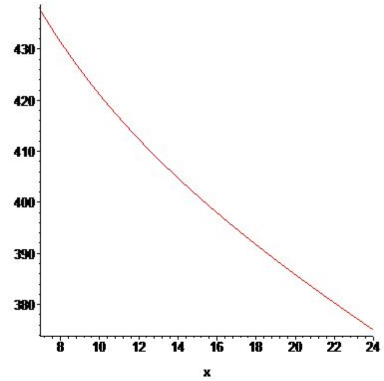
(iv) $f(0) = \alpha a + \beta d$, $f(d) = \alpha\sqrt{d^2 + a^2}$.

(v) In this case $x_0 = 105/8 \in (0, 24)$ so it is an acceptable critical point. We calculate $f(0) = 479$, $f(x_0) = 416$, and $f(24) = 425$. So the minimum cost is 416 TL.

(vi) In this case x_0 does not exist so we calculate $f(0) = 513$ and $f(24) = 375$. Thus the minimum cost is 375 TL.



$y = f(x)$ in **(v)**



$y = f(x)$ in **(vi)**

Remark: Note that $f''(x) = \dfrac{\alpha a}{(x^2 + a^2)^{3/2}} > 0$ so the graph of $y = f(x)$ is always concave up.

If $\alpha > \beta$, then $f'(x) = 0$ has a root, $x_0 = \dfrac{\beta a}{\sqrt{\alpha^2 - \beta^2}}$. Our function $f(x)$ decreases until $x = x_0$ and then increases. Therefore if $x_0 \in (0, d)$, then the minimum value of $f(x)$ is $f(x_0)$. Otherwise the minimum value is $f(d)$.

If however $\alpha \leq \beta$, then $f'(x) = 0$ has no root and the function is always decreasing. Hence the minimum value is always $f(d)$.