Date: 30 July 2001, Monday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

NAME:..

STUDENT NO:..

Math 102 Calculus II – Final Exam
Solution Manual

---

1) Calculate $\lim_{x\to 0} \dfrac{3\tan x^2 - 3x^2}{7x^6 + 8x^7}$.

Solution 1) First find the Taylor expansion of $\tan\theta$.
$f(\theta) = \tan\theta,\ f(0) = 0,$
$f'(\theta) = \sec^2\theta,\ f'(0) = 1,$
$f''(\theta) = 2\sec^2\theta\tan\theta,\ f''(0) = 0,$
$f'''(\theta) = 4\sec^2\theta\tan^2\theta + 2\sec^4\theta,\ f'''(0) = 2,$
so $\tan\theta = \theta + \frac{\theta^3}{3} +$ higher terms,
$3\tan x^2 = 3x^2 + x^6 +$ higher terms.
Hence $\dfrac{3\tan x^2 - 3x^2}{7x^6 + 8x^7} = \dfrac{x^6 + \text{higher terms}}{7x^6 + \text{higher terms}} \to \dfrac{1}{7}$ as $x \to 0$.

---

2) Find the maximum value of the function $f(x, y) = 5x + 2y + xy - x^2 - y^2$.

Solution 2)
$f_x = 5 + y - 2x = 0,\ f_y = 2 + x - 2y = 0$. The only solution is $(4, 3)$.
$f_{xx} = -2,\ f_{xy} = 1,\ f_{yy} = -2.$
$\Delta = f_{xx}f_{yy} - f_{xy}^2 = 3 > 0.$
Hence $(4, 3)$ is a local maximum point, but since it is the only critical point it must be the global maximum point. Thus the maximum value of the function is $f(4, 3) = 13$.

---

3) Calculate $\lim_{R\to\infty} I_R$, where $I_R = \displaystyle\int_0^R \int_{y^2}^{R^2} ye^{-x^2}\,dxdy$.

Solution 3)
Changing the order of integration we get $I_R = \displaystyle\int_0^{R^2}\int_0^{\sqrt{x}} ye^{-x^2}\,dydx = \int_0^{R^2} e^{-x^2}\left(\frac{1}{2}y^2\Big|_0^{\sqrt{x}}\right)dx =$
$\displaystyle\frac{1}{2}\int_0^{R^2} xe^{-x^2}\,dx = \frac{1}{4}\left(-e^{-x^2}\Big|_0^{R^2}\right) = \frac{1}{4}(1 - e^{-R^4}).$
So $\displaystyle\lim_{R\to\infty} I_R = \frac{1}{4}$.

---

4) Let S be the surface $z + x^2 + y^2 = 1$, $z \geq 0$, and $\vec{n}$ the unit normal vector of S pointing outwards. Consider the vector field $F = (-y + xz + z^2, x + yz^2 + z^3, z^7 - z)$. Calculate $\int\int_S \mathbf{curl} F \cdot \vec{n} d\sigma$.

Solution 4) The boundary of S is the unit circle in the xy-plane parametrized as $\vec{r}(t) = (\cos t, \sin t, 0)$, $0 \leq t \leq 2\Pi$.
$\mathbf{curl} F = \nabla \times F$, and Stokes' theorem says $\int\int_S \nabla \times F \cdot \vec{n} d\sigma = \oint_C F \cdot d\vec{r}$.
$F|C = (-\sin t, \cos t, 0)$.
$d\vec{r}(t) = (-\sin t, \cos t, 0)dt$
$F \cdot d\vec{r} = dt$.
So the required integral becomes $\oint_C F \cdot d\vec{r} = \int_0^{2\pi} dt = 2\pi$.

It is also possible to calculate the integral $\int\int_S \nabla \times F \cdot \vec{n} d\sigma$ directly. For this let $f = z + x^2 + y^2 - 1$.
$\nabla f = (2x, 2y, 1)$, $R = \{(x, y, 0) | x^2 + y^2 \leq 1\}$, $\mathbf{p} = \mathbf{k}$
$\mathbf{curl} F = (-2yz - 3z^2, x + 2z, 2)$
$\vec{n} = \frac{\nabla f}{|\nabla f|}$, $d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}$.
Hence $\mathbf{curl} F \cdot \vec{n} d\sigma = \mathbf{curl} F \cdot \nabla f dxdy$,
and the required integral becomes $\int\int_R \mathbf{curl} F \cdot \nabla f dxdy = \int\int_R [4y(1-x)(1-x^2-y^2) - 6x(1-x^2-y^2)^2 + 2xy + 2] dxdy = \int_0^{2\pi} \int_0^1 (2 + 2\cos\theta \sin\theta) r dr d\theta = 2\pi$.

---

5) Let K be the regular octagon shown in the figure. Let $\vec{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$.

Calculate the counterclockwise circulation, $\int_K \vec{F} \cdot \mathbf{T} ds$, of the vector field $\vec{F}$ around K.

Solution 5) This is a version of Example 6 on page 1092 of the book.
Let C be a circle of radius ϵ centered at the origin, where $0 < \epsilon < 1/2$. Orient C counterclockwise. Let E be the region between K and C. By direct calculation we find $\text{div} \vec{F} = 0$. By Green's theorem we have

$$\int_{K-C} \vec{F} \cdot \mathbf{T} ds = \int\int_E \text{div} \vec{F} dA = 0.$$

so $\int_K \vec{F} \cdot \mathbf{T} ds = \int_C \vec{F} \cdot \mathbf{T} ds$.
C is parametrized as $r(t) = (\epsilon \cos t, \epsilon \sin t)$, $0 \leq t \leq 2\pi$.
$dr = (-\epsilon \sin t, \epsilon \cos t)dt$, $\vec{F}|C = (-\frac{\sin t}{\epsilon}, \frac{\cos t}{\epsilon})$
$\vec{F}|C \cdot dr = dt$, so $\int_C \vec{F} \cdot \mathbf{T} ds = \int_0^{2\pi} dt = 2\pi$.

Hence $\int_K \vec{F} \cdot \mathbf{T} ds = 2\pi$.
