Date: 23 June 2001, Saturday

# Math 102 Calculus II – Midterm Exam I
# SOLUTIONS

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**1-a)** Calculate $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n$.

**Solution 1-a)** Let $a_n = (1+1/n^2)^n$. $\ln a_n = n\ln\left(1+\frac{1}{n^2}\right) = \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n}}$. Now applying L'Hopital's rule to this gives $\lim_{n\to\infty}\ln a_n = \lim_{x\to\infty}\frac{\frac{x^2}{1+x^2}\cdot\frac{-2}{x^3}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{2x^4}{x^5+x^3} = 0$. Since $\lim_{n\to\infty}\ln a_n = 0$, we get $\lim_{n\to\infty} a_n = 1$.

**1-b)** Check for convergence the series $\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^n n!$.

**Solution 1-b)** Let $a_n = \frac{2^n}{n^n}n!$. Apply the ratio test

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!} = \frac{2(n+1)n^n}{(n+1)^{n+1}}$$

$$= 2\left(\frac{n}{n+1}\right)^n = 2\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\right) \to \frac{2}{e} \text{ as } n\to\infty.$$

Since $\frac{2}{e} < 1$, the series converges by the ratio test.

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**2-a)** Check for convergence the series $\sum_{n=1}^{\infty}\frac{5\cdot 8\cdot 11\cdots(3n+5)}{3\cdot 5\cdot 7\cdots(2n+3)}\frac{2^n}{3^n}$.

**Solution 2-a)** Let $a_n = \frac{5\cdot 8\cdot 11\cdots(3n+5)}{3\cdot 5\cdot 7\cdots(2n+3)}\frac{2^n}{3^n}$. The ratio test fails. But observe that $a_{n+1}/a_n = \frac{6n+16}{6n+15} > 1$. So $\lim_{n\to\infty} a_n \neq 0$, and the series diverges by the divergence test.

**2-b)** Find the radius of convergence of the power series $\sum_{n=2}^{\infty}(n\ln n)x^n$. Check also the end points.

**Solution 2-b)** Let $a_n = (n\ln n)x^n$. Apply the ratio test; $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|$. For convergence we must have $|x| < 1$. When $|x| = 1$, the general term does not go to zero as $n$ goes to infinity, so the series diverges. So the interval of convergence is $-1 < x < 1$.

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**3)** Assume that $f(x)$ is an analytic function of $x$ with $f(0) = 0$ and $f'(x) = (1 - x^2)^{-2/3}$. Calculate the limit $\lim_{x \to 0} \dfrac{f(x) - x}{x^3}$.

**Solution 3)** The binomial theorem gives
$(1+t)^m = 1 + mt + \dfrac{m(m-1)}{2!}t^2 + \dfrac{m(m-1)(m-2)}{3!}t^3 + \cdots$
Putting $t = -x^2$, we get
$(1-x^2)^{2/3} = 1 + \dfrac{2}{3}x^2 - \dfrac{5}{9}x^4 + \cdots = f'(x)$. Integrating all sides term by term we get
$f(x) = x + \dfrac{2}{9}x^3 - \dfrac{1}{9}x^5 + \cdots$. From this we get
$\dfrac{f(x) - x}{x^3} = \dfrac{2}{9} - \dfrac{1}{9}x^2 + \cdots$. Hence $\lim_{x \to 0} \frac{f(x)-x}{x^3} = \frac{2}{9}$.

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**4)** Find a power series solution for $y' + y = 3x^2$ with $y(0) = 0$. Recognize the function in terms of elementary functions.

**Solution 4)** Since $y(0) = 0$, $y = a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Putting this into the above DE we get $3x^2 = a_1 + (a_1 + 2a_2)x + (a_2 + 3a_3)x^2 + \cdots + (a_n + (n+1)a_{n+1})x^n + \cdots$. From this we find $a_1 = 0$, $a_2 = 0$, $a_3 = 1$, $a_4 = -1/4$, $a_5 = 1/20$, and in general we claim that $a_n = (-1)^{n+1}3!/n!$, for $n \geq 3$. Putting this claim into the coefficient of $x^n$ we get, after simplifying, $a_{n+1} = (-1)^{n+2}3!/(n+1)!$, as claimed. This gives $y = 3! \sum_{n=3}^{\infty} (-1)^{n+1} \frac{x^n}{n!}$. Adding and subtracting $6 - 6x + 3x^2$ to this we get $y = 6 - 6x + 3x^2 - 6e^{-x}$.

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**5)** Calculate $\lim_{x \to 0} \dfrac{2 - x^4 - 2\cos x^2}{24x^8 + 8x^{24}}$.

Recall that $\cos x^2 = 1 - x^4/2 + x^8/24 - x^{12}/120 + \cdots$. Putting this into the above expression we get

$$\lim_{x \to 0} \frac{2 - x^4 - 2\cos x^2}{24x^8 + 8x^{24}} = \lim_{x \to 0} \frac{-x^8/12 + x^{12}/60 - \cdots}{24x^8 + 8x^{24}} = -\frac{1}{288}.$$

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