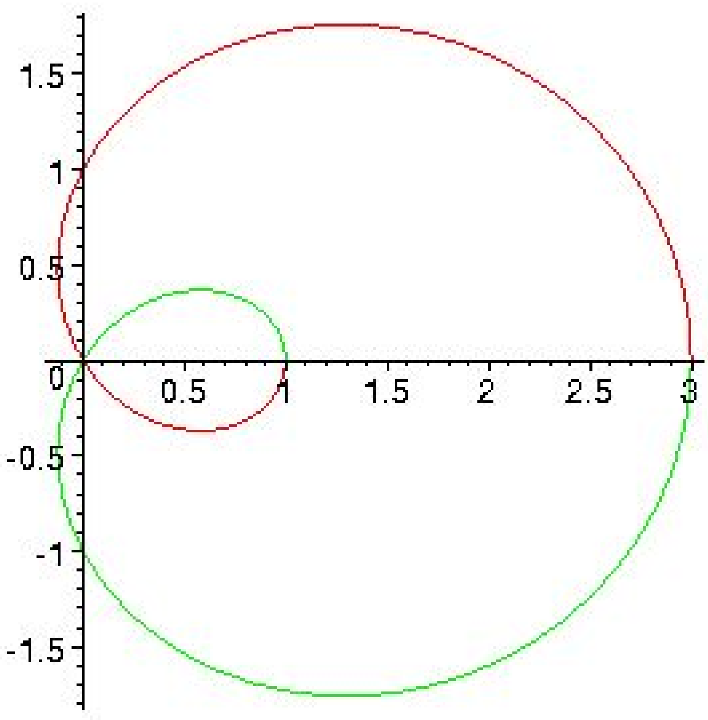
Date: 14 July 2001, Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 102 Calculus II – Midterm Exam II
SOLUTIONS

1) Plot the graph of $r = 1 + 2\cos(\theta)$, for $0 \leq \theta \leq 2\pi$.

Solution-1) The graph is symmetric about x-axis.
When $0 \leq \theta \leq \pi/2$, $\cos\theta$ decreases from 1 to 0, so r decreases from 3 to 1.
When $\pi/2 \leq \theta \leq 2\pi/3$, $\cos\theta$ decreases from 0 to $-1/2$, so r decreases from 1 to 0.
When $2\pi/3 \leq \theta \leq \pi$, $\cos\theta$ decreases from $-1/2$ to -1, so r decreases from 0 to -1.
Now flipping the resulting curve about x-axis gives the full graph.



2) Let ω be a differentiable function of x and y, and let $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions of t. Using the table below, calculate $\dfrac{d^2\omega}{dt^2}(0)$.

$f(0) = 1$	$f(1) = 0$	$g(0) = 1$	$g(1) = 0$
$f'(0) = 2$	$f'(1) = 3$	$g'(0) = 4$	$g'(1) = 6$
$f''(0) = -3$	$f''(1) = -2$	$g''(0) = 10$	$g''(1) = 11$

$\dfrac{\partial\omega}{\partial x}(0,0) = 31$	$\dfrac{\partial\omega}{\partial x}(1,1) = 13$	$\dfrac{\partial\omega}{\partial y}(0,0) = 18$	$\dfrac{\partial\omega}{\partial y}(1,1) = 8$	$\dfrac{\partial^2\omega}{\partial x\partial y}(0,0) = 3$
$\dfrac{\partial^2\omega}{\partial^2 x}(0,0) = 6$	$\dfrac{\partial^2\omega}{\partial^2 x}(1,1) = 7$	$\dfrac{\partial^2\omega}{\partial^2 y}(0,0) = 8$	$\dfrac{\partial^2\omega}{\partial^2 y}(1,1) = -6$	$\dfrac{\partial^2\omega}{\partial x\partial y}(1,1) = 5$

Solution-2) $\dfrac{d\omega}{dt}(0) = \dfrac{\partial\omega}{\partial x}(1,1)\cdot f'(0) + \dfrac{\partial\omega}{\partial y}(1,1)\cdot g'(0)$

$$\begin{aligned}
\frac{d^2\omega}{dt^2}(0) &= \left(\frac{\partial^2\omega}{\partial x^2}(1,1)\cdot f'(0) + \frac{\partial^2\omega}{\partial y\partial x}(1,1)\cdot g'(0)\right)\cdot f'(0) + \frac{\partial\omega}{\partial x}(1,1)\cdot f''(0) \\
&\quad + \left(\frac{\partial^2\omega}{\partial x\partial y}(1,1)\cdot f'(0) + \frac{\partial^2\omega}{\partial y^2}(1,1)\cdot g'(0)\right)\cdot g'(0) + \frac{\partial\omega}{\partial y}(1,1)\cdot g''(0) \\
&= ((7)(2)+(5)(4))\,(2) + (13)(-3) \\
&\quad + ((5)(2)+(-6)(4))\,(4) + (8)(10) \\
&= 53.
\end{aligned}$$

3) Let E be the tangent plane to the surface $3x^2 + 4y^2 - 2z^2 = 1$ at the point $(1, 2, 3)$. Let $\omega = x^2 + 8xy + 8y^3 + z^5$, subject to the condition that $(x, y, z) \in E$.

Calculate $\left(\frac{\partial \omega}{\partial x}\right)_z$ at the point $(x, z) = (1, -1)$.

Solution-3) The surface is given by $f(x, y, z) = 3x^2 + 4y^2 - 2z^2 - 1 = 0$. The gradient of f is $\nabla f = (6x, 8y, -4z)$. Evaluating at the point $(1, 2, 3)$ gives $\nabla f(1, 2, 3) = (6, 16, -12)$ which is the normal vector of the plane E. Thus the equation of E is $g(x, y, z) = 6 \cdot (x - 1) + 16 \cdot (y - 2) - 12 \cdot (z - 3) = 0$, or $g(x, y, z) = 3x + 8y - 6z - 1 = 0$.
Now differentiate ω with respect to x keeping in mind that z is free but y is dependent:

$$\left(\frac{\partial \omega}{\partial x}\right)_z (x, y, z) = 2x + 8y + 8x\frac{\partial y}{\partial x} + 24y^2\frac{\partial y}{\partial x}.$$

From the restraint $g = 0$ we get by differentiating both sides with respect to x, $3 + 8\frac{\partial y}{\partial x} = 0$, or $\frac{\partial y}{\partial x} = -\frac{3}{8}$.

From $g(1, y, -1) = 0$, we find $y = -1$. Finally substituting in these values we get

$$\left(\frac{\partial \omega}{\partial x}\right)_z (1, -1, -1) = 2 - 8 + 8(-3/8) + 24(-3/8) = -18.$$

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4) What is the largest value that the function $f(x, y) = 6xy - 4x^3 - 3y^2$ can take?

Solution-4) $f_x = 6y - 12x^2$, $f_y = 6x - 6y$. From $f_x = f_y = 0$ we find that the critical points are $(0, 0)$ and $(1/2, 1/2)$.
$f_{xx} = -24x, \quad f_{xy} = 6, \quad f_{yy} = -6.$
$\Delta = f_{xx}f_{yy} - (f_{xy})^2.$
At $(0, 0)$, $\Delta(0, 0) = -36 < 0$, so $(0, 0)$ is a saddle point.
At $(1/2, 1/2)$, $\Delta(1/2, 1/2) = 36 > 0$ and $f_{xx}(1/2, 1/2) = -12 < 0$, so $(1/2, 1/2)$ is a local maximum point. The value of f at this local maximum point is $f(1/2, 1/2) = 1/4$.
However, the function has neither global maximum nor global minimum values as can be seen by checking the limits
$\lim_{x\to\infty} f(x, 0) = -\infty$ and $\lim_{x\to-\infty} f(x, 0) = \infty$.

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5) Find the minimum and maximum values of $f(x, y) = x + 2y + 3$ subject to the condition that $4x^2 + 5y^2 = 84/5$.

Solution-5) Let $g(x, y) = 4x^2 + 5y^2 - 84/5$. $\nabla f = \lambda \nabla g$ gives $(1, 2) = \lambda(8x, 10y)$, or $x = 1/(8\lambda), \quad y = 1/(5\lambda)$. Using the constraint $g(1/(8\lambda), 1/(5\lambda)) = 0$, we find that $x = \pm 1$ and $y = \pm 8/5$.
Then the maximum value of f is $f(1, 8/5) = 36/5$, and the minimum value is $f(-1, -8/5) = -6/5$.