Date: 7 July 2003, Monday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 102 Calculus – Midterm Exam II
Solutions

Q-1) Find $\dfrac{dw}{dt}$ at $t = 1$, where

$$\begin{aligned} w(x,y) &= x^7e^y + \cos(xy) + y^3 \\ x(t) &= 2t^2 - \frac{t}{2} + \arctan t - (\frac{\pi}{4} - \frac{3}{2}) \\ y(t) &= t^4 + 2t + \arcsin \frac{t}{\sqrt{2}} - (3 + \frac{\pi}{4}). \end{aligned}$$

Solution:

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\ \frac{\partial w}{\partial x} &= 7x^6e^y - y\sin(xy) \\ \frac{\partial w}{\partial y} &= x^7e^y - x\sin(xy) + 3y^2 \\ x'(t) &= 4t - \frac{1}{2} + \frac{1}{1+t^2} \\ y'(t) &= 4t^3 + 2 + \frac{1}{\sqrt{1-(t^2/2)}}\frac{1}{\sqrt{2}} \\ x'(1) &= 4 \\ y'(1) &= 7 \\ x(1) &= 3 \\ y(1) &= 0 \\ \frac{\partial w}{\partial x}|_{t=1} &= \frac{\partial w}{\partial x}(3,0) = 7\cdot 3^6 \\ \frac{\partial w}{\partial y}|_{t=1} &= \frac{\partial w}{\partial y}(3,0) = 3^7 \\ \frac{dw}{dt}|_{t=1} &= (\frac{\partial w}{\partial x}|_{t=1})\, x'(1) + (\frac{\partial w}{\partial y}|_{t=1})\, y'(1) \\ &= (7\cdot 3^6)(4) + (3^7)(7) \\ &= (7\cdot 3^6)(4+3) \\ &= 7^2 3^6 \\ &= 35721. \end{aligned}$$

Q-2) Find $\left(\dfrac{\partial w}{\partial x}\right)_y$ at the point $(x, y, z) = (0, 0, 1)$, where

$$w = 3x + x^3 + 5y + y^5 + 7z + z^7 \tag{1}$$

and x, y, z are related by the equation

$$\ln(x^2 + z^2) + \ln(y^2 + z^2) + xyz - 8x = 0. \tag{2}$$

Solution:
Differentiating both sides of (1) with respect to x, treating y as constant, we get

$$\left(\frac{\partial w}{\partial x}\right)_y = 3(1 + x^2) + 7(1 + z^6)\frac{\partial z}{\partial x}. \tag{3}$$

Next differentiate both sides of (2) with respect to x, again treating y as constant, to get

$$\left(\frac{1}{x^2 + z^2}\right)(2x + 2z\frac{\partial z}{\partial x}) + \left(\frac{1}{y^2 + z^2}\right)(2z\frac{\partial z}{\partial x}) + yz + xy\frac{\partial z}{\partial x} - 8 = 0. \tag{4}$$

Put $(x, y, z) = (0, 0, 1)$ in (4) to get $\dfrac{\partial z}{\partial x} = 2$.

Finally putting $(x, y, z) = (0, 0, 1)$ and $\dfrac{\partial z}{\partial x} = 2$ in (3) we get

$$\left(\frac{\partial w}{\partial x}\right)_y = 31 \text{ at } (0, 0, 1).$$

Q-3) Write the equation of the tangent plane and the normal line to the surface $f(x, y, z) = 0$ at the point $(x, y, z) = (1, 2, -1)$, where

$$f(x, y, z) = x^3 + x^2y + xz^2 + y^3 + yz + z^5 - 9.$$

Solution:

$$\begin{aligned} f_x &= 3x^2 + 2xy + z^2, & f_x(1, 2, -1) &= 8, \\ f_y &= x^2 + 3y^2 + z, & f_y(1, 2, -1) &= 12, \\ f_z &= 2xz + y + 5z^4, & f_z(1, 2, -1) &= 5. \end{aligned}$$

The equation of the tangent plane is

$$8(x - 1) + 12(y - 2) + 5(z + 1) = 0, \qquad \text{or equivalently}$$
$$8x + 12y + 5z = 27.$$

Normal line is given by the parametric equations,
$x = 1 + 8t$,
$y = 2 + 12t$,
$z = -1 + 5t$, $t \in \mathbb{R}$.

Q-4) Let

$$f(x,y) = x^2 + 4xy + 5y^2 + 6x - 2y.$$

Find the critical points of f and determine if they give minimum, maximum or saddle points. If f has a global minimum or maximum value calculate it explicitly.

Solution:
Solving

$$f_x = 2x + 4y + 6 = 0, \quad \text{and} \quad f_y = 4x + 10y - 2 = 0$$

we find that $(x, y) = (-17, 7)$ is the only critical point.

Next apply the second derivative test:
$f_{xx} = 2 > 0,\ f_{yy} = 10,\ f_{xy} = 4$, and
$\nabla = f_{xx}f_{yy} - f_{xy}^2 = 4 > 0$.
Hence $(-17, 7)$ is a minimum point. Since this is the only critical point, this gives the global minimum.

The global minimum value of f can now be easily calculated:

$$\begin{aligned} f(x,y) &= x(x+2y+6) + y(2x+5y-2) \\ f(-17,7) &= (-17)(-17+14+6) + (7)(-34+35-2) \\ &= (-17)(3) + (7)(-1) \\ &= -51 - 7 \\ &= -58. \end{aligned}$$

Q-5) Consider the curve obtained by intersecting the paraboloid surface

$$x^2 + y^2 - z = 25 \text{ with the vertical plane } y = 3.$$

Find the points on this curve which are closest to the origin.

Solution:
We want to minimize the square of the distance function:

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraints

$$g_1(x,y,z) = 0 \text{ and } g_2(x,y,z) = 0$$

where

$$g_1 = x^2 + y^2 - z - 25 \text{ and } g_2 = y - 3.$$

Use Lagrange multipliers method:

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2,$$

which gives

$$\begin{aligned} 2x &= 2\lambda x, \\ 2y &= 2\lambda y + \mu, \\ 2z &= -\lambda. \end{aligned}$$

The first equation implies that either $\lambda = 1$ or else $x = 0$.
Case 1: Assume $\lambda = 1$. Then from the third equation $z = -1/2$. From $g_2 = 0$ we have $y = 3$. Putting these into $g_1 = 0$ we get $x = \pm\sqrt{31/2}$.

The critical points supplied by this case are $\left(\pm\sqrt{\frac{31}{2}}, 3, -\frac{1}{2}\right)$.

Case 2: Assume $\lambda \neq 1$. Then from the first equation we get $x = 0$. From $g_2 = 0$ we get $y = 3$. Putting these into $g_1 = 0$ we get $z = -16$.
The critical point supplied by this case is $(0, 3, -16)$.

Evaluating f at these critical points we find
$f\left(\pm\sqrt{\frac{31}{2}}, 3, -\frac{1}{2}\right) = \frac{99}{4}$,
$f(0, 3, -16) = 265$.

Hence the points on this curve nearest the origin are $\left(\pm\sqrt{\frac{31}{2}}, 3, -\frac{1}{2}\right)$.
