Math 102 Calculus II – Quiz-5
Solutions

18 July, 2003 Ali Sinan Sertöz

1-a) Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)}$.

Solution: Letting

$$a_n = \frac{1}{(n-1)(n+1)}$$

and using partial fractions technique we get

$$2a_n = \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}.$$

Adding the terms of the series we find

$$\begin{aligned}
2a_2 &= 1 - \frac{1}{3} \\
2a_3 &= \frac{1}{2} - \frac{1}{4} \\
2a_4 &= \frac{1}{3} - \frac{1}{5} \\
&\vdots \\
2a_{n-2} &= \frac{1}{n-3} - \frac{1}{n-1} \\
2a_{n-1} &= \frac{1}{n-2} - \frac{1}{n} \\
2a_n &= \frac{1}{n-1} - \frac{1}{n+1}
\end{aligned}$$

from which it follows that the partial sum

$$S_n = a_1 + \cdots a_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right)$$

and after taking the limit as $n \to \infty$ we find that the sum is

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}.$$

1-b) Test for convergence $\displaystyle\sum_{n=1}^{\infty} \frac{2^n \ln n}{n!}$.

Solution: Let $a_n = \frac{2^n \ln n}{n!}$. Using the ratio test we find that

$$\frac{a_{n+1}}{a_n} = \frac{2\ln(n+1)}{(n+1)\ln n}$$

which converges to zero as $n \to \infty$. Hence the series converges by the Ratio Test.

2-a) Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

Solution: Letting

$$a_n = \frac{1}{n(n+3)}$$

and using partial fractions technique we get

$$3a_n = \frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}.$$

Adding the terms of the series we find

$$\begin{aligned}
3a_1 &= 1 - \frac{1}{4} \\
3a_2 &= \frac{1}{2} - \frac{1}{5} \\
3a_3 &= \frac{1}{3} - \frac{1}{6} \\
3a_4 &= \frac{1}{4} - \frac{1}{7} \\
&\vdots \\
3a_{n-3} &= \frac{1}{n-3} - \frac{1}{n} \\
3a_{n-2} &= \frac{1}{n-2} - \frac{1}{n+1} \\
3a_{n-1} &= \frac{1}{n-1} - \frac{1}{n+2} \\
3a_n &= \frac{1}{n} - \frac{1}{n+3}
\end{aligned}$$

from which it follows that the partial sum

$$S_n = a_1 + \cdots a_n = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

and after taking the limit as $n \to \infty$ we find that the sum is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{18}.$$

2-b) Test for convergence $\sum_{n=0}^{\infty} \frac{n!}{2^{n^2}}$.

Solution: Let $a_n = n!/2^{n^2}$. Using the ratio test we find that

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n^2+n+1}}$$

which converges to zero as $n \to \infty$. Hence the series converges by the Ratio Test.