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Math 102 Calculus II – Final Exam – Solutions

Q-1) Test the following series for convergence:

i) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^{15}}$ **ii)** $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(1+n^2)(\ln n)^2}$

Solution: **i)** $\left|\dfrac{a_{n+1}}{a_n}\right| = 2\left(\dfrac{n}{n+1}\right)^{15} \to 2$ as $n \to \infty$. The series diverges by ratio test.

ii) Let $a_n = \dfrac{n}{(1+n^2)(\ln n)^2}$ and $b_n = \dfrac{1}{n(\ln n)^2}$.

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1$, so $\sum a_n$ converges if and only if $\sum b_n$ converges. On the other hand

$\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2}$ converges, so $\sum b_n$ converges by the integral test.

Hence $\sum a_n$ converges.

Q-2) Find the critical points of $f(x,y) = 27x^3 + y^3 - 12xy + 7$, and decide if each critical point is a local min/max or a saddle point. Find global min/max points, if they exist.

Solution: From $f_y = 3y^2 - 12x = 0$, we have $x = y^2/4$. Putting this into $f_x = 81x^2 - 12y = 0$ we get $y(27y^3 - 64) = 0$; $y = 0$ or $y = 4/3$.

The critical points are $(0,0)$ and $(4/9, 4/3)$.

Second derivative test: $f_{xx} = 162x,\ f_{yy} = 6y,\ f_{xy} = -12,\ \Delta = 972xy - 144$.

$\Delta(0,0) < 0$, so $(0,0)$ is a saddle point.

$\Delta(4/9, 4/3) > 0,\ f_{xx}(4/9, 4/3) > 0$, so $(4/9, 4/3)$ is a local min point.

Since $f(0,y) = y^3 + 7$, there are no global min or max.

Q-3) Evaluate $\int_0^1 \int_y^1 \int_0^{z^2} \cos x \; dx \; dz \; dy$.

Solution: $\int_0^1 \int_y^1 \int_0^{z^2} \cos x \; dx \; dz \; dy = \int_0^1 \int_y^1 \sin z^2 \; dz \; dy = \int_0^1 \int_0^z \sin z^2 \; dy \; dz = \int_0^1 z \sin z^2 \; dz =$ $-\frac{1}{2} \cos z^2 \Big|_0^1 = \frac{1}{2}(1 - \cos 1)$.

Q-4) Find the volume of the region lying inside the sphere $x^2 + y^2 + z^2 = 20$ and inside the cylinder $x^2 + y^2 = 16$ but outside the paraboloid $z = x^2 + y^2$.

Solution: The sphere $x^2 + y^2 + z^2 = 20$ and the paraboloid $z = x^2 + y^2$ intersect when $z + z^2 = 20$ or $z = 0$. On the xy-plane the projection of this intersection is the circle $x^2 + y^2 = 4$.

The sphere $x^2 + y^2 + z^2 = 20$ and the cylinder $x^2 + y^2 = 16$ intersect when $z = 2$, and the projection of this intersection on the xy-plane is $x^2 + y^2 = 16$. We can now set up the volume integral in cylindrical coordinates:

$$\begin{aligned} Volume &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{20-r^2}}^{r^2} r \; dz \; dr \; d\theta + \int_0^{2\pi} \int_2^4 \int_{-\sqrt{20-r^2}}^{\sqrt{20-r^2}} r \; dz \; dr \; d\theta \\ &= \int_0^{2\pi} \int_0^2 (r^3 - r\sqrt{20 - r^2}) \; dr \; d\theta + \int_0^{2\pi} \int_2^4 2r\sqrt{20 - r^2} \; dr \; d\theta \\ &= (2\pi) \left(\frac{1}{4} r^4 - \frac{1}{3}(20 - r^2)^{3/2} \Big|_0^2 \right) + (2\pi) \left(-\frac{2}{3}(20 - r^2)^{3/2} \Big|_2^4 \right) \\ &= \frac{40\pi}{3}(3 + 2\sqrt{5}). \end{aligned}$$

Q-5) Evaluate the circulation $\oint_C \mathbf{F} \cdot \mathbf{T} \; ds$ of the vector field

$\mathbf{F} = \left(\frac{2x^2}{\sqrt{1+x^2}} \arctan \frac{y}{\sqrt{1+x^2}}, \; x \ln(1 + x^2 + y^2) \right)$ around the curve C, where C is given by $x^2 + y^2 = 3$ and is oriented counterclockwise.

Solution: Let $\mathbf{r} = (x, y)$ be a parametrization of the circle and R the region inside. Let $\mathbf{F} = (M, N)$.

$\oint_C \mathbf{F} \cdot \mathbf{T} \; ds = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (M dx + N dy) = \int\int_R (N_x - M_y) dx dy = \int\int_R \ln(1 + x^2 + y^2) \; dx dy =$ $\int_0^{2\pi} \int_0^{\sqrt{3}} r \ln(1 + r^2) \; dr \; d\theta = \frac{1}{2} \int_0^{2\pi} \int_1^4 \ln(u) \; du \; d\theta = \frac{1}{2}(2\pi) \left(u \ln u - u \Big|_1^4 \right) = (8 \ln 2 - 3)\pi$.
