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MATH 102 MIDTERM II Solutions

1) Find the limit if it exists, or show that the limit does not exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}$ **(b)** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

Solution-a: $x^2y^2 \le 2x^2y^2+x^4+y^4 \le (x^2+y^2)^2$, so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} \le \lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)^2}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} x^2+y^2 = 0$. Hence $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = 0$ by the sandwich theorem.

Solution-b: Let $y = \lambda x^2$, then $\dfrac{x^2y}{x^4+y^2} = \dfrac{\lambda x^4}{x^4+\lambda^2x^4} = \dfrac{\lambda}{1+\lambda^2}$ when $x \ne 0$. Hence the limit depends on how (x,y) approaches the origin, and we say that the limit does not exist.

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2. Let $f(x,y) = \tan\left(\pi\sin(\dfrac{\pi}{4\sqrt{3}}\,y) - \dfrac{\pi}{3}x^{12}\right)$ and $x(t) = \tan t$, $y(t) = \sec\dfrac{2t}{3}$.

If we set $g(t) = f\,(x(t), y(t))$, find $g'\left(\dfrac{\pi}{4}\right)$.

Solution: Let $h(x,y) = \pi\sin(\frac{\pi}{4\sqrt{3}}\,y) - \frac{\pi}{3}x^{12}$.

Then $f(x,y) = \tan(h(x,y)$, and $g(t) = \tan\left(h(x(t),y(t))\right)$.

By the chain rule $g'(t) = \sec^2\left(h(x(t),y(t))\right)\ [h_x(x(t),y(t))\ x'(t) + h_y(x(t),y(t))\ y'(t)]$.

Finally $g'(\frac{\pi}{4}) = \sec^2\left(h(x(\frac{\pi}{4}),y(\frac{\pi}{4}))\right)\ \left[h_x(x(\frac{\pi}{4}),y(\frac{\pi}{4}))\ x'(\frac{\pi}{4}) + h_y(x(\frac{\pi}{4}),y(\frac{\pi}{4}))\ y'(\frac{\pi}{4})\right]$.

Note that: $x'(t) = \sec^2 t$, $y'(t) = \dfrac{2}{3}\sec\dfrac{2t}{3}\tan\dfrac{2t}{3}$,

$h_x(x,y) = -4\pi x^{11}$, $h_y = \dfrac{\pi^2}{4\sqrt{3}}\cos(\dfrac{\pi}{4\sqrt{3}}\,y)$.

$x(\dfrac{\pi}{4}) = 1$, $y(\dfrac{\pi}{4}) = \dfrac{2}{\sqrt{3}}$,

$x'(\dfrac{\pi}{4}) = 2$, $y'(\dfrac{\pi}{4}) = \dfrac{4}{9}$

$h_x(1,\dfrac{2}{\sqrt{3}}) = -4\pi$, $h_y(1,\dfrac{2}{\sqrt{3}}) = \dfrac{\pi^2}{8}$.

Putting these together we find, $g'(\dfrac{\pi}{4}) = \dfrac{2\pi(\pi-144)}{27}$

3. Let $f(x, y) = \ln(1 + x^2 + y^2)$.

(a) Find ∇f.

(b) Find the directional derivative of f at the point $(1, 2)$ in the direction pointing from $(1, 2)$ towards the point $(4, 6)$.

(c) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(3, 2)$.

(d) Find the parametric equations of a normal line to the surface $z = f(x, y)$ at the point $(0, 0)$.

Solution-a: $\nabla f = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2} \right)$.

Solution-b: $v = (4, 6) - (1, 2) = (3, 4)$, $\vec{u} = (3/5, 4/5)$, $\nabla f(1, 2) = (1/3, 2/3)$. $D_{\vec{u}} f(1, 2) = \nabla f \cdot \vec{u} = 11/5$.

Solution-c: $\nabla f(3, 2) = (3/7, 2/7)$, $f(3, 2) = \ln 14$. Equation of the tangent plane at $(3, 2)$ is $\nabla f(3, 2) \cdot (x - 3, y - 2) - (z - \ln 14) = 0$, or after simplifying $3x + 2y - 7z = 13 - 7 \ln 14$.

Solution-d: $\nabla f(0, 0) = (0, 0)$ so an equation of a normal line will be $x = 0, \; y = 0, \; z = t$, where $t \in \mathbb{R}$.

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4. Let $f(x, y) = y(1 + x) + \ln \frac{1}{x^2 y^3}$, where $x, y > 0$.
Find the global minimum, maximum and saddle points of f, if they exist, in the given domain.

Solution:

$f(x, y) = y + xy - 2 \ln x - 3 \ln y$, $f_x = y - 2/x$, $f_y = 1 + x - 3/y$. The only critical point is $(2, 1)$. $f_{xx}(2, 1) > 0$, and $f_{xx}(2, 1) f_{yy}(2, 1) - f_{xy}^2(2, 1) > 0$ so this critical point is a local minimum, but since it is the only critical point it must be the global minimum.

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5. Find the extreme values of $f(x, y, z) = 4x^2 + y^2 + z^2$ subject to the condition $x^4 - y^2z^2 = \dfrac{9}{4}$. For each extreme value decide if it is a minimum or a maximum value.

Solution: $\Delta f = (8x, 2y, 2z)$, $g(x, y, z) = x^4 - y^2z^2 - 9/4$, $\Delta g = (4x^3, -2yz^2, -2y^2z)$.

$\Delta f = \lambda \Delta g$ gives:

(1) $8x = 4\lambda x^3$,
(2) $2y = -2\lambda yz^2$,
(3) $2z = -2\lambda y^2 z$.

From (1) $2x = \lambda x^3$.

Case-1: $x = 0$.
Then $g(0, y, z) < 0$, contradiction.

Case-2: $x \neq 0$.
Then $x^2 = 2/\lambda$, so in particular $\lambda > 0$.

Subcase 2.1: $y = 0$.
Then from (3), $z = 0$. $g(x, 0, 0) = 0$ gives $x = \pm\sqrt{3/2}$, so $f(\pm\sqrt{3/2}, 0, 0) = 6$.

Subcase 2.2: $y \neq 0$.
Then from (2) $z^2 = -1/\lambda$ which is a contradiction since $\lambda > 0$ in case-2.

So the only critical value of f is 6. Letting $y = z = t$ and solving x as $\pm(9/4 + t^4)^{1/4}$ from $g = 0$ we see that $f(\pm(9/4 + t^4)^{1/4}, t, t) = 4\sqrt{9/4 + t^2} + 2t^2$ and that this is unbounded as t becomes large. Hence 6 is the global minimum value of f.
