Date: 24 July 2006, Monday NAME:...
Instructor: Ali Sinan Sertöz
Time: 15:30-17:30 STUDENT NO:...

Math 102 Calculus – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ: Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in detail.

Q-1) Find all values of x for which the power series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n\,3^n}\, x^n$ converges.

Solution: Letting $a_n = \frac{\ln n}{n\,3^n}\, x^n$, we use the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n}{n+1}\,\frac{\ln(n+1)}{\ln n}\,\frac{|x|}{3} \to \frac{|x|}{3} \text{ as } n \to \infty.$$

Therefore the series converges absolutely for all $|x| < 3$.

We check the end points separately.

When $x = 3$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$, and diverges by direct comparison with the harmonic series, $0 < \dfrac{1}{n} < \dfrac{\ln n}{n}$ for all $n \geq 3$.

When $x = -3$, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$. Letting $f(x) = \ln(x)/x$, we find that $f'(x) = (1 - \ln x)/x^2$ which is negative for $x > e$. It now follows that the alternating series test applies to our series and it converges.

Hence the interval of convergence is $[-3, 3)$.

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Q-2) We have $w = w(x, y)$, i.e. w is a function of x and y. We also know that $x = x(r, s)$, $y = y(s, t)$, $r = r(u)$, $s = s(u, v)$, $t = t(v)$.
The following equations are known to hold at $(u, v) = (0, 1)$:
$t = 3, s = 5, r = 7, y = 9, x = 11$,
$t_v = 2, s_u = 4, s_v = 6, r_u = 8$,
$y_s = \pi, y_t = e, x_r = -1, x_s = -2$,
$w_x = 10, w_y = -10$.
a: Write w_u using chain rule.
b: Find w_u at $(u, v) = (0, 1)$.

Solution:

$$w_u = w_x(x_r r_u + x_s s_u) + w_y y_s s_u.$$

Putting in the above values we find that $w_u = -160 - 40\pi$ at $(u, v) = (0, 1)$.

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Q-3) Let $f(x,y) = \frac{1}{3}x^3 - \frac{9}{2}x^2 + 7x - xy + \frac{1}{2}y^2 + 9y + 1.$

Find all local min/max and saddle points of f. *(15 points)*
Does f have global min/max points? *(5 points)*

Solution: $f_x = x^2 - 9x + 7 - y = 0$ and $f_y = -x + y + 9 = 0$ gives $(8, -1)$ and $(2, -7)$ as the critical points.

$f_{xx} = 2x - 9$, $f_{xy} = -1$, $f_{yy} = 1$ and $\Delta = 2x - 10$.

Using the second derivative test we find that $(8, -1)$ is a local min point and $(2, -7)$ is a saddle point.

Keeping $y = 0$ and varying x we see that f goes both to $= \infty$ and $-\infty$, so the function has no global min or max.

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Q-4) Evaluate the triple integral $\int\int\int_R f(x,y,z)\ dV$ where $f(x,y,z) = y$, and R is the region in the first octant enclosed by the surfaces $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$, $x, y, z \geq 0$.

Solution: First we find that the projection of the curve of intersection of these surfaces is $x^2 + y^2 = 1$. Then we start evaluating the integral.

$$\begin{aligned}
\int\int\int_R f(x,y,z)\ dV &= \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} y\ dzdydx \\
&= 2\int_0^1 \int_0^{\sqrt{1-x^2}} [(1-x^2)y - y^3]\ dydx \\
&= \frac{1}{2}\int_0^1 (1-x^2)^2\ dx \\
&= \frac{4}{15}.
\end{aligned}$$

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Q-5) Find the flux of the vector field $\mathbf{F} = \left(\frac{x^3}{3}\right)\mathbf{i} + \left(\frac{x^3+3xz+z^3}{z^2+2}\right)\mathbf{j} + \left(\frac{x^2+2xy-y^2}{y^4+2}\right)\mathbf{k}$ across the sphere S given by $x^2+y^2+z^2=4$ along the outward unit normal vector $\mathbf{n}$. i.e. evaluate the integral

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

Solution: Trying to evaluate this integral directly is too tiring. We try the divergence theorem which says

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_D \nabla\cdot\mathbf{F}\,dV$$

where D is the solid sphere which is inside of S. $\nabla\cdot\mathbf{F} = x^2$. Passing to spherical coordinates we find

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \int\int\int_D \nabla\cdot\mathbf{F}\,dV \\
&= \int\int\int_D x^2\,dx\,dy\,dz \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (\rho\sin\phi\cos\theta)^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_0^{2}\rho^4\,d\rho\right) \\
&= \left(\frac{4}{3}\right)(\pi)\left(\frac{32}{5}\right) \\
&= \frac{128\pi}{15}.
\end{aligned}$$