NAME: STUDENT NO:

Q-3) Among all rectangular regions $0 \leq x \leq a$, $0 \leq y \leq b$, find the one for which the total outward flux

$$\oint_C \mathbf{F} \cdot \vec{\mathbf{n}}\, ds$$

of $\mathbf{F} = \left(\frac{2}{3}x^3 y\right)\, \mathbf{i} + \left(2xy^3 - 17xy^2\right)\, \mathbf{j}$ across the four sides is least. Here C denotes the boundary of the given rectangle with the positive orientation.

Solution: The Green's theorem in the plane says that if $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, then

$$\oint_C \mathbf{F} \cdot \vec{\mathbf{n}}\, ds = \int\int_{R_{ab}} (M_x + N_y)\ dxdy$$

where R_{ab} is the given rectangle with corners at the points $(0,0)$, $(a,0)$, (a,b) and $(0,b)$. We can calculate easily the above double integral;

$$\begin{aligned} \int_0^b \int_0^a \left(2x^2y + 6xy^2 - 34xy\right)\ dxdy &= \int_0^b \left(\frac{2}{3}a^3y + 3a^2y^2 - 17a^2y\right)\ dxdy \\ &= \frac{1}{3}a^3b^2 + a^2b^3 - \frac{17}{2}a^2b^2. \end{aligned}$$

We now must minimize $f(a,b) = \dfrac{1}{3}a^3b^2 + a^2b^3 - \dfrac{17}{2}a^2b^2$ where $a, b \geq 0$. On the boundary f is zero, so we look for interior critical points.

$f_a = a^2b^2 + 2ab^3 - 17ab^2 = ab^2(a + 2b - 17) = 0$,
$f_b = \frac{2}{3}a^3b + 3a^2b^2 - 17a^2b = \frac{1}{3}a^2b(2a + 9b - 51) = 0$.
This gives $(a,b) = \left(\dfrac{51}{5}, \dfrac{17}{5}\right)$ as the only interior critical point.

We calculate easily that $f(\dfrac{51}{5}, \dfrac{17}{5}) = -\dfrac{51^3 \cdot 17^2}{5^5 \cdot 6} < 0$. Since f is zero on the boundary and goes to infinity as a and b go to infinity, this critical point gives the global minimum.

Hence the required size of the rectangle giving the minimal flux is $a = \frac{51}{5}$ and $b = \frac{17}{5}$.