Math 102 Calculus II – Homework I

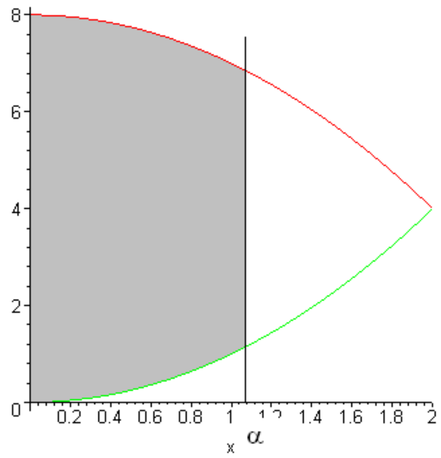
Due on July 6, 2007 Friday 17:00

Q-1) For $0 < \alpha < 2$, define

$$F(\alpha) = \int_0^{\alpha^2}\int_0^{\sqrt{y}} f\ dxdy + \int_{\alpha^2}^{8-\alpha^2}\int_0^{\alpha} f\ dxdy + \int_{8-\alpha^2}^{8}\int_0^{\sqrt{8-y}} f\ dxdy$$

where $f = \dfrac{y\sin x}{4-x^2}\cdot$ Evaluate $F(\alpha)$ explicitly in terms of α.

Solution: The region of integration is the shaded region of the following figure.

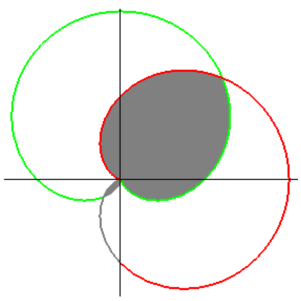


Changing the order of integration on this region we find

$$F(\alpha) = \int_0^{\alpha}\int_{x^2}^{8-x^2} \frac{\sin x}{4-x^2}\, y\ dydx = 8\int_0^{\alpha} \sin x\ dx = 8(1-\cos\alpha).$$

Q-2) Find the area of the region common to the cardioids $r = 1+\sin\theta$ and $r = 1+\cos\theta$.

Solution: The two cardioids intersect as follows:



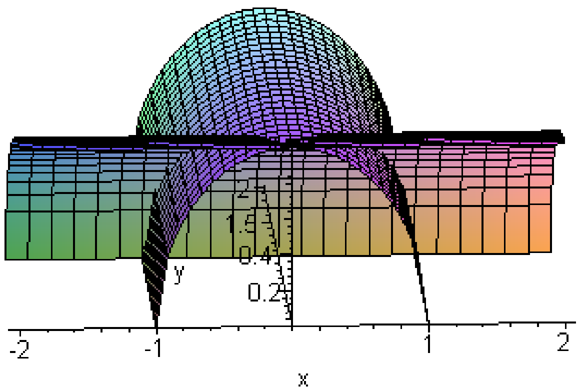
The common area can be found as:

$$\int_{-3\pi/4}^{\pi/4}\int_0^{1+\sin\theta} r\ drd\theta + \int_{\pi/4}^{5\pi/4}\int_0^{1+\cos\theta} r\ drd\theta = 2(\frac{3\pi}{4} - \sqrt{2}) \approx 1.88.$$

Q-3) Let $F(a)$ denote the volume of the region common to the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = a^2$, where $a \geq 1$. Write the integral expression for $F(a)$. Evaluate $F(1)$ explicitly. Using a computer software find a such that $F(a) = 2F(1)$.

Solution:

Two cylinders of the same radii in general intersect as follows:



In our case we find

$$F(a) = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{a^2 - x^2}\, dydx = 8 \int_0^1 \sqrt{(1-x^2)(a^2-x^2)}\, dx.$$

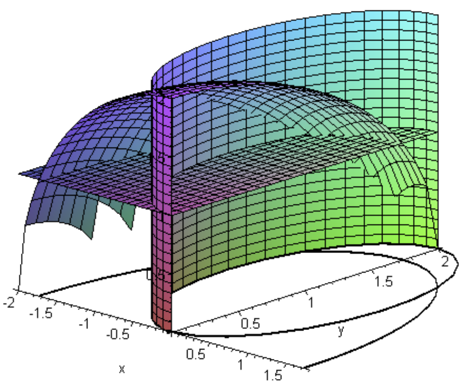
We easily find $F(1) = \dfrac{16}{3}$.

It turns out that if $a = \sqrt{3.143} \approx 1.77$, then $F(a) \approx 2F(1)$.

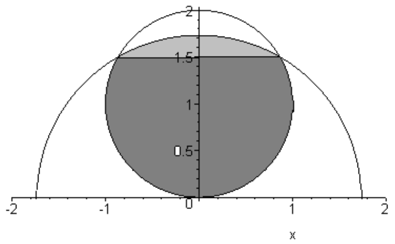
Q-4) Find the volume of the region bounded from above by $x^2 + y^2 + z^2 = 4$, from below by $z = 1$ and from the sides by $x^2 + y^2 - 2y = 0$.

Solution:

The region is as follows.



The shadow of the $z = 1$ base of the region in the xy-plane is as follows.



Note that the semicircle here is $x^2 + y^2 = 3$ obtained by putting $z = 1$ in the sphere equation.

Using the symmetry we set up the volume integral as

$$2\int_0^{3/2}\int_0^{\sqrt{2y-y^2}}\int_1^{\sqrt{4-x^2-y^2}} dzdxdy + 2\int_{3/2}^{\sqrt{3}}\int_0^{\sqrt{3-y^2}}\int_1^{\sqrt{4-x^2-y^2}} dzdxdy.$$

Changing to cylindrical coordinates

$$2\int_0^{\pi/3}\int_0^{2\sin\theta}\int_1^{\sqrt{4-r^2}} rdzdrd\theta + 2\int_{\pi/3}^{\pi/2}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} rdzdrd\theta$$

and evaluating we find the first integral as $2\left(\frac{5\pi}{9} - \frac{3\sqrt{3}}{4}\right)$, and the second integral as $2\left(\frac{5\pi}{36}\right)$.

Hence the volume is $\frac{25\pi}{18} - \frac{3\sqrt{3}}{2} \approx 1.76$.

Q-5) For $n \geq 2$, let V_n denote the *volume* of the region

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}.$$

For example $V_2 = \pi$ and $V_3 = 4\pi/3$. Find V_4 and V_5.

Solution:

Let $V_n(R)$ denote the *volume* of the region

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq R^2\}.$$

Note that

$$V_4 = V_4(1) = \int_{-1}^{1}\boxed{\int_{-\sqrt{1-x_4^2}}^{\sqrt{1-x_4^2}}\int_{-\sqrt{1-x_4^2-x_3^2}}^{\sqrt{1-x_4^2-x_3^2}}\int_{-\sqrt{1-x_4^2-x_3^2-x_2^2}}^{\sqrt{1-x_4^2-x_3^2-x_2^2}} dx_1dx_2dx_3}\, dx_4$$

and the value in the box is precisely $V_3(\sqrt{1-x_4^2})$ for which we have a formula,

$$V_3(R) = \frac{4\pi}{3}R^3, \text{ so } V_3(\sqrt{1-x_4^2}) = \frac{4\pi}{3}(1-x_4^2)^{3/2}.$$

Hence

$$V_4 = \int_{-1}^{1} V_3(\sqrt{1-x_4^2})\,dx_4 = \left(\frac{4\pi}{3}\right)\int_{-1}^{1}(1-x_4^2)^{3/2}\,dx_4 = \left(\frac{3\pi}{8}\right)\left(\frac{4\pi}{3}\right) = \frac{\pi^2}{2},$$

where we evaluate the integral with the substitution $x_4 = \sin\theta$.

A similar line of argument gives $V_4(R^2) = \dfrac{\pi^2}{2}R^4$ which we need to calculate V_5.

Observe that

$$\begin{aligned}
V_5 &= V_5(1) \\
&= \int_{-1}^{1} \boxed{\int_{-\sqrt{1-x_5^2}}^{\sqrt{1-x_5^2}} \int_{-\sqrt{1-x_5^2-x_4^2}}^{\sqrt{1-x_5^2-x_4^2}} \int_{-\sqrt{1-x_5^2-x_4^2-x_3^2}}^{\sqrt{1-x_5^2-x_4^2-x_3^2}} \int_{-\sqrt{1-x_5^2-x_4^2-x_3^2-x_2^2}}^{\sqrt{1-x_5^2-x_4^2-x_3^2-x_2^2}} dx_1 dx_2 dx_3 dx_4}\, dx_5 \\
&= \int_{-1}^{1} V_4(\sqrt{1-x_5^2})\,dx_5 \\
&= \frac{\pi^2}{2}\int_{-1}^{1}(1-x_5^2)^2\,dx_5 = \frac{\pi^2}{2}\cdot\frac{16}{15} \\
V_5 &= \frac{8\pi^2}{15}.
\end{aligned}$$

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