Q-5) For $n \geq 2$, let V_n denote the *volume* of the region

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}.$$

For example $V_2 = \pi$ and $V_3 = 4\pi/3$. Find V_4 and V_5.

Solution:

Let $V_n(R)$ denote the *volume* of the region

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq R^2\}.$$

Note that

$$V_4 = V_4(1) = \int_{-1}^{1} \boxed{\int_{-\sqrt{1-x_4^2}}^{\sqrt{1-x_4^2}} \int_{-\sqrt{1-x_4^2-x_3^2}}^{\sqrt{1-x_4^2-x_3^2}} \int_{-\sqrt{1-x_4^2-x_3^2-x_2^2}}^{\sqrt{1-x_4^2-x_3^2-x_2^2}} dx_1 dx_2 dx_3} \, dx_4$$

and the value in the box is precisely $V_3(\sqrt{1-x_4^2})$ for which we have a formula,

$$V_3(R) = \frac{4\pi}{3} R^3, \text{ so } V_3(\sqrt{1-x_4^2}) = \frac{4\pi}{3}(1-x_4^2)^{3/2}.$$

Hence

$$V_4 = \int_{-1}^{1} V_3(\sqrt{1-x_4^2}) \, dx_4 = \left(\frac{4\pi}{3}\right) \int_{-1}^{1} (1-x_4^2)^{3/2} \, dx_4 = \left(\frac{3\pi}{8}\right)\left(\frac{4\pi}{3}\right) = \frac{\pi^2}{2},$$

where we evaluate the integral with the substitution $x_4 = \sin\theta$.

A similar line of argument gives $V_4(R^2) = \dfrac{\pi^2}{2} R^4$ which we need to calculate V_5.

Observe that

$$\begin{aligned}
V_5 &= V_5(1) \\
&= \int_{-1}^{1} \boxed{\int_{-\sqrt{1-x_5^2}}^{-\sqrt{1-x_5^2}} \int_{-\sqrt{1-x_5^2-x_4^2}}^{\sqrt{1-x_5^2-x_4^2}} \int_{-\sqrt{1-x_5^2-x_4^2-x_3^2}}^{\sqrt{1-x_5^2-x_4^2-x_3^2}} \int_{-\sqrt{1-x_5^2-x_4^2-x_3^2-x_2^2}}^{\sqrt{1-x_5^2-x_4^2-x_3^2-x_2^2}} dx_1 dx_2 dx_3 dx_4} \, dx_5 \\
&= \int_{-1}^{1} V_4(\sqrt{1-x_5^2}) \, dx_5 \\
&= \frac{\pi^2}{2} \int_{-1}^{1} (1-x_5^2)^2 \, dx_5 = \frac{\pi^2}{2} \cdot \frac{16}{15} \\
V_5 &= \frac{8\pi^2}{15}.
\end{aligned}$$

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