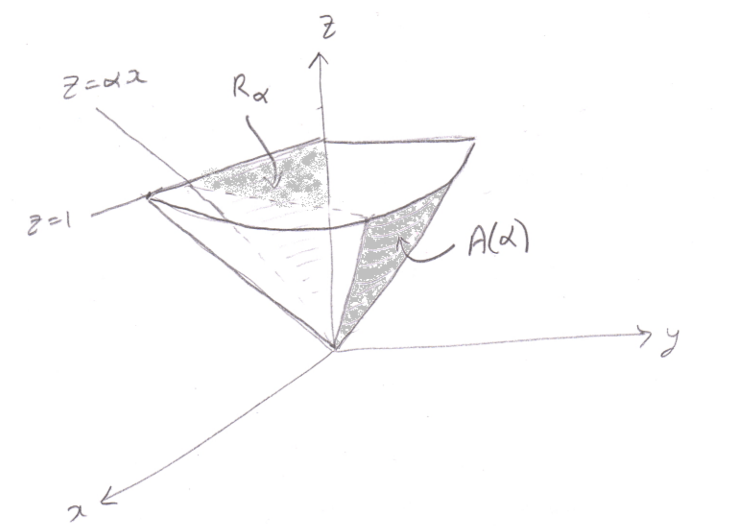
Q-4) Let R_α be the region in the xz-plane bounded by the lines $z = \alpha x$, $z = 1$ and $x = 0$, where $\alpha \geq 1$. Let $A(\alpha)$ denote the area of the surface $z^2 = x^2 + y^2$ lying above R_α. First, without doing any calculations, find $A(1)$ and $\lim_{\alpha\to\infty} A(\alpha)$. Then calculate $A(\alpha)$ explicitly in terms of α. Check your answer with what you found above.

Solution:



$A(1)$ is the surface area of the cone lying in the first quadrant and below the plane $z = 1$, which is $\pi/(2\sqrt{2})$. When α goes to infinity, the line $z = \alpha x$ becomes the z-axis and then we have no area, giving $\lim_{\alpha\to\infty} A(\alpha) = 0$.

We now calculate the surface area of the cone over the region R_α. Here $f = x^2 + y^2 - z^2$, $\nabla f = (2x, 2y, -2z)$, $|\nabla f| = 2\sqrt{2}z$, $p = (0, 1, 0)$.

$$\begin{aligned}
A(\alpha) &= \int\int_{R_\alpha} \frac{|\nabla f|}{|\nabla f \cdot p|}\, dA \\
&= \sqrt{2}\int_0^{1/\alpha}\int_{\alpha x}^{1} \frac{z}{\sqrt{z^2 - x^2}}\, dz dx \\
&= \sqrt{2}\int_0^{1/\alpha}\left(\sqrt{z^2 - x^2}\Big|_{z=\alpha x}^{z=1}\right) dx \\
&= \sqrt{2}\int_0^{1/\alpha}(\sqrt{1 - x^2} - \sqrt{\alpha^2 - 1}\, x)\, dx \\
&= \sqrt{2}\left(\frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2}\arcsin x - \frac{\sqrt{\alpha^2 - 1}}{2}x^2\Big|_0^{1/\alpha}\right) \\
&= \frac{1}{\sqrt{2}}\arcsin\frac{1}{\alpha}.
\end{aligned}$$