Date: June 28, 2007, Thursday NAME:..
Time: 9:30-11:30
Ali Sinan Sertöz STUDENT NO:..

Math 102 Calculus II – Midterm Exam II

1	2	3	4	5	TOTAL
10	30	10	20	30	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Q-1) Find $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$, if the limit exists. If it does not exit, then explain why .

Solution:

$$\frac{x^3y}{x^6+y^2} = \frac{\lambda}{1+\lambda^2}$$

when $y = \lambda x^3$, so the limit does not exist.

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Q-2) Let $f(x, y, z) = x^2 + xy + yz$, $x(s, t) = s^2 - st + 2t + 2s - 2$, $y(s, t) = t^2 + s^2 t + s - 8$, $z(s, t) = s^2 t - 1$. Find the directional derivative of $h(s, t) = f(x(s, t), y(s, t), z(s, t))$ at the point $(s, t) = (1, 2)$ in the direction of the vector $\vec{u} = (3, 4)$.

Solution: $h_s = \nabla f \cdot (x_s, y_s, z_s)$, $h_t = \nabla f \cdot (x_t, y_t, z_t)$.
$\nabla f = (2x + y, x + z, y)$.
$x(1, 2) = 3$, $y(1, 2) = -1$, $z(1, 2) = 1$, so $\nabla f(3, -1, 1) = (5, 4, -1)$.
$x_s = 2s - t + 2 = 2$ at $(s, t) = (1, 2)$.
$y_s = 2st + 1 = 5$ at $(s, t) = (1, 2)$.
$z_s = 2st = 4$ at $(s, t) = (1, 2)$.

$h_s = (5, 4, -1) \cdot (2, 5, 4) = 26$.

$x_t = -s + 2 = 1$ at $(s, t) = (1, 2)$.
$y_t = 2t + s^2 = 5$ at $(s, t) = (1, 2)$.
$z_t = s^2 = 1$ at $(s, t) = (1, 2)$.

$h_t = (5, 4, -1) \cdot (1, 5, 1) = 24$.

$\nabla h(1, 2) = (26, 24)$, $\nabla h(1, 2) \cdot (3, 4) = (26, 24) \cdot (3, 4) = 174$, $|\vec{u}| = 5$

Finally we have $D_{\vec{u}} h(1, 2) = \frac{1}{5} \nabla h(1, 2) \cdot (3, 4) = \frac{174}{5}$.

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Q-3) Find an equation for the tangent line of the plane curve $x^2+xy+y^3=11$ at the point $(1,2)$. Is the point $(4,1)$ on this tangent line?

Solution: Let $f(x,y)=x^2+xy+y^3-11$. Check that $f(1,2)=0$ so the given point is on the curve. $\nabla f=(2x+y,x+3y^2)$. $\nabla f(1,2)=(4,13)$.

An equation for the tangent line at $(1,2)$ is $(4,13)\cdot(x-1,y-2)=0$ or equivalently

$$4x+13y=30.$$

Check that $4\cdot 4+13\cdot 1=29$ so the point $(4,1)$ is not on this line.

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Q-4) Find and classify all the critical points of the function $f(x,y) = x^3 - 2xy^2 - x + 2y^2$.

Solution:
$f_x = 3x^2 - 2y^2 - 1 = 0$
$f_y = -4xy + 4y = 4y(1-x) = 0$

Case 1: $y = 0$**.** From $f_x = 0$ we get $x = \pm 1/\sqrt{3}$. The critical points in this case are $(\pm 1/\sqrt{3}, 0)$.

Case 2: $y \neq 0$. Then $x = 1$ and from $f_x = 0$ we get $y = \pm 1$. The critical points of this case are $(1, \pm 1)$.

$f_{xx} = 6x$, $f_{yy} = 4(1-x)$, $f_{xy} = -4y$, $\Delta = 8[3x(1-x) - 2y^2]$.

At $(1/\sqrt{3}, 0)$, $\Delta > 0$, $f_{xx} > 0$, so this is a local minimum point.

At the other critical points $\Delta < 0$, so they are all saddle points.

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Q-5) Find the minimum and the maximum values of the function $f(x,y,z) = x^2 - y^3 + 3z^2$ on the sphere $x^2 + y^2 + z^2 = 25$.

Solution: Let $g(x,y,z) = x^2+y^2+z^2-25$. $\nabla f = \lambda \nabla g$ gives $(2x, -3y^2, 6z) = \lambda(x,y,z)$, or:
$x(\lambda - 2) = 0$
$y(\lambda + 3y) = 0$
$z(\lambda - 6) = 0$.

Case 1: $x = 0$.
Case 1.1: $y = 0$. Then $g(0,0,z) = 0$ gives $z = \pm 5$, $\boxed{f(0,0,\pm 5) = 75}$.
Case 1.2: $y \neq 0$. Then $y = -\lambda/3$.
Case 1.2.1: $z = 0$. From $g(0,y,0) = 0$ we get $y = \pm 5$. $\boxed{f(0,5,0) = -125}$, $\boxed{f(0,-5,0) = 125}$.
Case 1.2.2: $z \neq 0$. Then $\lambda = 6$ and hence $y = -2$. From $g(0,-2,z) = 0$ we get $z = \pm\sqrt{21}$. $\boxed{f(0,-2,\pm\sqrt{21}) = 71}$.
Case 2: $x \neq 0$. Then $\lambda = 2$ and this forces $z = 0$.
Case 2.1: $y = 0$. Then $g(x,0,0) = 0$ gives $x = \pm\sqrt{5}$. $\boxed{f(\pm\sqrt{5},0,0) = 25}$.
Case 2.2: $y \neq 0$. Then $y = -\lambda/3 = -2/3$. $g(x,-2/3,0) = 0$ gives $x = \pm\sqrt{25 - 4/9}$.
$\boxed{f(\pm\sqrt{25-4/9}, -2/3, 0) = 25 - 4/27}$.

Thus we see that the maximal value is 125 and the minimal value is -125.