Q-1) For any ϵ with $0 < \epsilon < \pi/6$, define the region R_ϵ as the region in $\mathbb{R}^2$ bounded by the curves $y = 1/x$, $y = 2/x$, $x = \epsilon$ and $x = \pi/6$. Calculate

$$\lim_{\epsilon \to 0} \int\int_{R_\epsilon} x^2 \sec^2(x^2 y)\, dx dy.$$

Solution:

$$\begin{aligned}
\int\int_{R_\epsilon} x^2 \sec^2(x^2 y)\, dx dy &= \int_\epsilon^{\pi/6} \int_{1/x}^{2/x} x^2 \sec^2(x^2 y)\, dy dx \\
&= \int_\epsilon^{\pi/6} \left[\tan(x^2 y) \Big|_{1/x}^{2/x} \right] dx \\
&= \int_\epsilon^{\pi/6} (\tan(2x) - \tan(x))\ dx \\
&= \left[-\frac{1}{2} \ln\cos 2x + \ln\cos x \Big|_\epsilon^{\pi/6} \right] \\
&= \frac{1}{2} \ln\frac{3}{2} - [-\frac{1}{2} \ln\cos 2\epsilon + \ln\cos\epsilon].
\end{aligned}$$

And since ln and cos are continuous functions,

$$\lim_{\epsilon \to 0} \int\int_{R_\epsilon} x^2 \sec^2(x^2 y)\, dx dy = \frac{1}{2} \ln\frac{3}{2}.$$