Q-5) Evaluate the integral

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

where S is the level surface given by $x^2 + z^2 - 4 + y^4 = 0,\ y \geq 0,$

$$\mathbf{F} = \left(x^2 z + \ln(y^2+1),\ \cosh(x^2+y^2) - \ln(z^2+1),\ \frac{y^3}{y^2+1} - xz^2 \right),$$

and $\mathbf{n}$ is the unit normal of S pointing out.

Solution: Let $D = \{(x,z) \in \mathbb{R}^2 \mid x^2 + z^2 \leq 4\}$ and let $C = \partial D$. Then using Stokes' theorem twice, we find that

$$\begin{aligned}\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds \\ &= \int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1}\, d\sigma\end{aligned}$$

where $\mathbf{n_1}$ is the unit normal of D pointing towards y-direction to be compatible with the orientation on C which in turn is induced by $\mathbf{n}$. Thus $\mathbf{n_1} = \mathbf{j}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n_1} = x^2 + z^2$. This gives

$$\begin{aligned}\int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1}\, d\sigma &= \int\int_D (x^2 + z^2)\, dx dz \\ &= \int_0^{2\pi} \int_0^2 r^3\, dr d\theta \\ &= (2\pi)(\frac{16}{4}) = 8\pi.\end{aligned}$$