Q-5) Evaluate the integral

$$\int\int_R \sin^2\left(\frac{x+y}{x-y}\right)\,dA$$

where R is the convex quadrilateral region with vertices at the points $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$.

Solution: First apply the change of coordinates with $u = x + y$ and $v = x - y$. Then the integral becomes

$$\begin{aligned}
\int\int_R \sin^2\left(\frac{x+y}{x-y}\right)\,dA &= \frac{1}{2}\int_1^2\int_{-v}^{v} \sin^2\left(\frac{u}{v}\right)\,dudv \\
&= \frac{1}{2}\int_1^2\int_{-v}^{v} \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2u}{v}\right)\right)\,dudv \\
&= \frac{3}{4} - \frac{3}{8}\sin 2 \approx 0.409.
\end{aligned}$$