Q-2) let $\mathbf{F} = \left(\frac{2x}{x^2+y^4}, \frac{4y^3}{x^2+y^4} \right)$. Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where C is the circle of radius R centered at the origin. Beware here that the Green's theorem does not hold since $\mathbf{F}$ is not defined at the origin. Observe that in this problem $M_y = N_x$ for the vector field $\mathbf{F} = (\mathbf{M}, \mathbf{N})$. Suppose you have the task of providing such vector fields on demand. How would you construct such vector fields without much effort? How did I *invent* the above vector field?

Solution: Let R_ϵ be the rectangle formed by the lines $x = \pm\epsilon$ and $y = \pm\epsilon$ where $\epsilon > 0$ is small enough that the rectangle R_ϵ totally lies inside C. Then Green's theorem applies to the region bounded by C and R_ϵ and since $M_y = N_x$, where $\mathbf{F} = (M, N)$, we must have

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_{R_\epsilon} \mathbf{F}\cdot\mathbf{T}\,ds$$

where R_ϵ is positively oriented. But using the obvious parametrization for each side of R_ϵ, we find that $\int_{R_\epsilon} \mathbf{F}\cdot\mathbf{T}\,ds = 0$ since on each side we are integrating an odd function on $[-1, 1]$. This gives

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = 0.$$

To generate such vector fields, start with any function and calculate its gradient. For example the above vector field is the gradient of $\ln(x^2+y^4)$. If the function is not defined at the origin, then its gradient also fails to be defined there.