Q-4) Evaluate the integral

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

where S is the level surface given by $x^2 + z^2 - 4(x+z) - y + 8 = 0$, $0 \le y \le 4$, and

$$\mathbf{F} = \left(x^2 z + \ln(y^2+1),\ \cosh(x^2+y^2) - \ln(z^2+1),\ \frac{y^3}{y^2+1} - xz^2 \right).$$

Solution: Let $D = \{(x,z) \in \mathbb{R}^2 \mid (x-2)^2 + (z-2)^2 \le 4\ \}$ and let the boundary of D be C. Then applying Stokes' theorem twice we get

$$\begin{aligned} \int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_C \mathbf{F} \cdot \mathbf{T}\, ds \\ &= \int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1}\, d\sigma \end{aligned}$$

where $\mathbf{n_1}$ is the unit normal of D pointing towards y-direction to be compatible with the orientation on C which in turn is induced by $\mathbf{n}$. Thus $\mathbf{n_1} = \mathbf{j}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n_1} = x^2 + z^2$. This gives

$$\begin{aligned} \int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1}\, d\sigma &= \int\int_D (x^2 + z^2)\, dxdz \\ &= \int_0^4 \int_{2-\sqrt{4z-z^2}}^{2+\sqrt{4z-z^2}} (x^2+z^2)\, dxdz \\ &= 40\pi, \end{aligned}$$

where the last line should be obtained through a computer algebra system.

An easy way to evaluate this integral by hand is to make the change of variables $X = x - 2$ and $Z = z - 2$, and change to polar coordinates in the new XZ system. This gives the easy integral

$$\int_0^{2\pi} \int_0^2 (r^2 + 4r(\cos\theta + \sin\theta) + 8) r dr d\theta = 40\pi.$$

Another way to solve this problem is to use Stokes' theorem only once. Then we parameterize the boundary of S as

$$\mathbf{r}(\theta) = (2 + 2\sin\theta,\ 2,\ 2 + 2\cos\theta),\quad \theta \in [0, 2\pi].$$

Notice how the correct orientation of the boundary is provided by the parametrization. Now we have

$$d\mathbf{r} = (2\cos\theta,\ 0,\ -2\sin\theta)\ d\theta,$$

and $\mathbf{F}\cdot d\mathbf{r}$ then becomes

$$(2\cos\theta)\left((2+2\sin\theta)^2(2+2\cos\theta)+\ln 17\right)+(-2\sin\theta)\left(\frac{64}{17}+(2+2\sin\theta)(2+2\cos\theta)^2\right).$$

Integrating this we get

$$\int_0^{2\pi} \mathbf{F}\cdot d\mathbf{r} = 40\pi.$$

Finally, you may try to evaluate the integral as is. Then you will get

$$\nabla\times\mathbf{F} = [3\,\frac{y^2}{y^2+1} - 2\,\frac{y^4}{(y^2+1)^2} + 2\,\frac{z}{z^2+1}, x^2+z^2, 2\,\sinh\left(x^2+y^2\right)x - 2\,\frac{y}{y^2+1}].$$

If $f = x^2+z^2-4(x+z)-y+8$, then the unit outward normal of S is in the direction of $-\nabla f$ where

$$\nabla f = [2\,x-4, -1, 2\,z-4].$$

You will have

$$\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \nabla\times\mathbf{F}\cdot\nabla f dxdz$$

and the integral will be evaluated over the disk $(x-2)^2+(z-2)^2\le 4$. You will also need to substitute y with $x^2+z^2-4(x+z)+8$ since the integrand lives on the surface S.

Putting these together, you will end up with the double integral

$$\int_0^4\int_{2-\sqrt{4z-z^2}}^{2+\sqrt{4z-z^2}} \left\{\left[3\,\frac{\left(x^2+z^2-4\,x-4\,z+8\right)^2}{\left(x^2+z^2-4\,x-4\,z+8\right)^2+1} - 2\,\frac{\left(x^2+z^2-4\,x-4\,z+8\right)^4}{\left(\left(x^2+z^2-4\,x-4\,z+8\right)^2+1\right)^2}\right.\right.$$
$$\left.+2\,\frac{z}{z^2+1}\right](-2\,x+4)+x^2+z^2$$
$$+\left[2\,\sinh\left(x^2+\left(x^2+z^2-4\,x-4\,z+8\right)^2\right)x\right.$$
$$\left.\left.-2\,\frac{x^2+z^2-4\,x-4\,z+8}{\left(x^2+z^2-4\,x-4\,z+8\right)^2+1}\right](-2\,z+4)\right\}\ dxdz.$$

An attempt to numerically evaluate this on Maple will give, after a long pause, 125 which is almost $40\pi\approx 125.6$.