Date: July 26, 2010, Monday NAME:..
Time: 15:00-17:00
Ali Sinan Sertöz STUDENT NO:..

Math 102 Calculus II – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	15	25	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail, if that is necessary. A correct answer without proper reasoning may not get any credit, except in the fill-in-the-blanks type problems.

Q-1) Find the minimum and maximum values of the function

$$f(x, y) = x^2 + y^2 - 2x + 2y + 5$$

on the closed disk

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}.$$

Solution: We first find the critical values of the function.
$f_x = 2x - 2 = 0$ gives $x = 1$.
$f_y = 2y + 2 = 0$ gives $y = -1$.
This critical point $(1, -1)$ is in the domain D, so we calculate f there; $\boxed{f(1, -1) = 3.}$

Next we restrict f to the boundary of D which is parameterized as $x = 2\cos t$, $y = 2\sin t$, $t \in [0, 2\pi]$. We then have

$$\phi(t) = f(2\cos t, 2\sin t) = 4\sin t - 4\cos t + 9, \; t \in [0, 2\pi].$$

We find its critical points;
$\phi'(t) = 4\sin t + 4\cos t = 0$, $\tan t = -1$, $t = 3\pi/4$ or $7\pi/4$ both in $[0, 2\pi]$.

These give the values:
$\boxed{g(3\pi/4) = f(-\sqrt{2}, \sqrt{2}) = 9 + 4\sqrt{2} \approx 15}$.
$\boxed{g(7\pi/4) = f(\sqrt{2}, -\sqrt{2}) = 9 - 4\sqrt{2} \approx 3.3}$.

Checking the values in the boxes we find that the maximum value of f is $9 + 4\sqrt{2}$ and the minimum value is 3.

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Q-2 Evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+x^3}\, dx\, dy$.

Solution:

$$
\begin{aligned}
\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+x^3}\, dx\, dy &= \int_0^2 \int_0^{x^2} \frac{1}{1+x^3}\, dy\, dx \\
&= \int_0^2 \left(\left. \frac{y}{1+x^3} \right|_0^{x^2} \right) dx \\
&= \int_0^2 \frac{x^2}{1+x^3}\, dx \\
&= \left(\left. \frac{1}{3}\ln(1+x^3) \right|_0^2 \right) \\
&= \frac{1}{3}\ln 9 \\
&= \frac{2}{3}\ln 3.
\end{aligned}
$$

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Q-3) Change the order of integration of the following integral as indicated and also find the value of the integral by evaluating any of the integrals you find.
(Grading: each box=1 point, evaluation=7 points.)

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{\sqrt{x^2+z^2}}^{3} z\, dy\, dx\, dz = \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} z\, dy\, dz\, dx$$

$$= \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} z\, dz\, dx\, dy + \int_{\square}^{\square} \int_{\square}^{\square} \int_{\square}^{\square} z\, dz\, dx\, dy.$$

Solution:

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{\sqrt{x^2+z^2}}^{3} z\, dy\, dx\, dz = \int_{\boxed{-1}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{\sqrt{1-x^2}}} \int_{\boxed{\sqrt{x^2+z^2}}}^{\boxed{3}} z\, dy\, dz\, dx$$

$$= \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{-y}}^{\boxed{y}} \int_{\boxed{0}}^{\boxed{\sqrt{y^2-x^2}}} z\, dz\, dx\, dy +$$

$$\int_{\boxed{1}}^{\boxed{3}} \int_{\boxed{-1}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{\sqrt{1-x^2}}} z\, dz\, dx\, dy.$$

The last integral is easier to evaluate and we find $\frac{1}{6} + \frac{4}{3} = \frac{3}{2}$.

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Q-4) Let R be the region in the plane bounded by $y = x^2$, $y = x + 2$, and $x = 0$. Let C be the boundary of R taken counterclockwise.

Let $F = (e^x + y^2 - \tan x + 1,\ \ln(y^3 + 1) + 2xy + x^3 - 7)$.

Calculate the work done by F along C, i.e. calculate $\int_C F \cdot T\, ds$.

Solution: Let $F = (M, N)$.

$$\begin{aligned}
\int_C F \cdot T\, ds &= \int_C M\, dx + N\, dy \\
&= \iint_R (N_x - M_y)\, dA \quad \text{(Green's Theorem)} \\
&= 3 \iint x^2\, dA \\
&= 3 \int_0^2 \int_{x^2}^{x+2} x^2\, dy\, dx \\
&= \frac{44}{5}.
\end{aligned}$$

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Q-5) Find the area of the region which lies in the plane $2x + 4y - z = 0$ and is bounded by the curve of intersection of this plane by the paraboloid $x^2 + y^2 - z = 9$.

Solution:

We want to calculate $d\sigma$ for the surface $F(x, y, z) = 2x + 4y - z = 0$.

The projection of the region on the xy-plane is given by $2x + 4y = x^2 + y^2 - 9$, which gives

$$(x-1)^2 + (y-2)^2 = 14.$$

Call this projection R.

$\nabla F = (2, 4, -1)$.
$|\nabla F| = \sqrt{4 + 16 + 1} = \sqrt{21}$.
$|\nabla F \cdot k| = 1$
Hence $d\sigma = \dfrac{|\nabla F|}{|\nabla F \cdot k|}\, dA = \sqrt{21}\, dA$.

Finally

$$\begin{aligned} \text{Area} &= \iint_R d\sigma \\ &= \sqrt{21} \iint_R dA \\ &= \sqrt{21} \cdot (\text{Area of } R) \\ &= \sqrt{21}\; 14\pi. \end{aligned}$$