Q-1) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^5 + y^6}{(x^2+y^2)^\alpha} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find all value of $\alpha \in \mathbb{R}$ such that both $f_x(0,0)$ and $f_y(0,0)$ exist. Calculate $f_x(0,0)$ and $f_y(0,0)$ for all such values of α.

Solution:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} x^{4-2\alpha} = \begin{cases} 1 & \text{if } \alpha = 2, \\ 0 & \text{if } \alpha < 2, \\ \text{No Limit} & \text{if } \alpha > 2. \end{cases}$$

Similarly,

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} y^{5-2\alpha} = \begin{cases} 1 & \text{if } \alpha = 5/2, \\ 0 & \text{if } \alpha < 5/2, \\ \text{No Limit} & \text{if } \alpha > 5/2. \end{cases}$$

Both limits exist if and only if $\alpha \leq 2$.

If $\alpha = 2$, then $f_x(0,0) = 1,\ f_y(0,0) = 0$.

If $\alpha < 2$, then $f_x(0,0) = 0,\ f_y(0,0) = 0$.