Q-4) Let $F(x) = \int_{x^4}^{x^3} \sqrt{t^3 + x^2}\, dt$. Calculate $F'(x)$ and find explicitly the values of $F'(0)$ and $F'(1)$.

Hint: Assume that you can differentiate under the integral sign; see the last few problems at the end of the section on "The Chain Rule" of Thomas' Calculus.

Solution:

Let $G(u, v, w) = \int_{u}^{v} \sqrt{t^3 + w}\, dt$ where u, v and w are functions of x.

Then

$$\frac{d}{dx} G(u, v, w) = G_u(u, v, w)u' + G_v(u, v, w)v' + G_w(u, v, w)w', \tag{1}$$

where

$$G_u = -\sqrt{u^3 + w}, \quad G_v = \sqrt{v^3 + w}, \quad G_w = \int_{u}^{v} \frac{1}{2\sqrt{t^3 + w}}\, dt. \tag{2}$$

Put $u = x^4$, $v = x^3$ and $w = x^2$ into the equations (**??**) and (**??**) to get

$$F(x) = G(x^4, x^3, x^2),$$

and

$$F'(x) = -4x^3\sqrt{x^{12} + x^2} + 3x^2\sqrt{x^9 + x^2} + \int_{x^4}^{x^3} \frac{x}{\sqrt{t^3 + x^2}}\, dt.$$

From this we immediately find

$$F'(0) = 0 \text{ and } F'(1) = -\sqrt{2}.$$