Q-1) Consider the function

$$f(x, y) = \begin{cases} 
\frac{x^5 + y^6}{(x^2 + y^2)^\alpha} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}$$

Find all value of $\alpha \in \mathbb{R}$ such that both $f_x(0, 0)$ and $f_y(0, 0)$ exist. Calculate $f_x(0, 0)$ and $f_y(0, 0)$ for all such values of $\alpha$.

Solution:

$$f_x(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \to 0} x^{4-2\alpha} = \begin{cases} 
1 & \text{if } \alpha = 2, \\
0 & \text{if } \alpha < 2, \\
\text{No Limit} & \text{if } \alpha > 2.
\end{cases}$$

Similarly,

$$f_y(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \to 0} y^{5-2\alpha} = \begin{cases} 
1 & \text{if } \alpha = 5/2, \\
0 & \text{if } \alpha < 5/2, \\
\text{No Limit} & \text{if } \alpha > 5/2.
\end{cases}$$

Both limits exist if and only if $\alpha \leq 2$.

If $\alpha = 2$, then $f_x(0, 0) = 1$, $f_y(0, 0) = 0$.

If $\alpha < 2$, then $f_x(0, 0) = 0$, $f_y(0, 0) = 0$. 
Q-2) Assume that \(3z + x + y^2 + xz^3 = 13\) defines \(z\) as a \(C^2\) function of \(x\) and \(y\) around the point \((x, y, z) = (3, 2, 1)\). Find the values of \(z_x\), \(z_y\), \(z_{xy}\), \(z_{yx}\), \(z_{xx}\) and \(z_{yy}\) at the point \((x, y, z) = (3, 2, 1)\).

Solution:

Differentiate both sides of \(3z + x + y^2 + xz^3 = 13\) implicitly with respect to \(x\), taking \(y\) as the other independent variable and \(z\) as a differentiable function of \(x\) and \(y\), to get

\[
3z_x + 1 + z^3 + 3xz^2z_x = 0,
\]

which gives

\[
z_x = -\frac{1 + z^3}{3(1 + xz^2)}.
\]

Putting \((x, y, z) = (3, 2, 1)\) into the equation (1) or (2), we get

\[
z_x = -\frac{1}{6}.
\]

Similarly we get

\[
z_y = -\frac{2y}{3(1 + xz^2)},
\]

and

\[
z_y = -\frac{1}{3}.
\]

Now using any of the equations (1),(2) or (3), differentiating implicitly and putting \((x, y, z, z_x, z_y) = (3, 2, 1, -1/6, -1/3)\), we get

\[
z_{xx} = \frac{1}{24}, \quad z_{yy} = -\frac{1}{3}, \quad z_{xy} = z_{yx} = 0.
\]
Q-3) Let $S$ be the surface in $\mathbb{R}^3$ given by $f(x, y, z) = 0$ where $f(x, y, z) = 1 + x^2 + y^4 - z$. Let $p_0 = (1/2, y_0, z_0)$ be a point on the surface such that the tangent plane to the surface $S$ at $p_0$ passes through the origin. Find $z_0$.

**Solution:**

\[ \nabla f(x, y, z) = (2x, 4y^3, -1). \]
\[ \nabla f(p_0) = (1, 4y_0^3, -1). \]

Equation of the tangent plane to $S$ at $p_0$ is

\[ (1, 4y_0^3, -1) \cdot (x - \frac{1}{2}, y - y_0, z - z_0) = 0. \]

This passes through the origin so $(x, y, z) = (0, 0, 0)$ satisfies this equation giving

\[ -\frac{1}{2} - 4y_0^4 + z_0 = 0. \]

(4)

Since $p_0 = (1/2, y_0, z_0) = 0$ is on the surface, we also have

\[ \frac{5}{4} + y_0^4 - z_0 = 0. \]

(5)

Adding equations (4) and (5) we get

\[ y_0^4 = \frac{1}{4}. \]

Putting this value into equation (4) or (5) we get

\[ z_0 = \frac{3}{2}. \]
Q-4) Let $F(x) = \int_{x^4}^{x^3} \sqrt{t^3 + x^2} \, dt$. Calculate $F'(x)$ and find explicitly the values of $F'(0)$ and $F'(1)$.

**Hint:** Assume that you can differentiate under the integral sign; see the last few problems at the end of the section on “The Chain Rule” of Thomas’ Calculus.

**Solution:**

Let $G(u, v, w) = \int_u^v \sqrt{t^3 + w} \, dt$ where $u$, $v$ and $w$ are functions of $x$.

Then

$$\frac{d}{dx} G(u, v, w) = G_u(u, v, w)u' + G_v(u, v, w)v' + G_w(u, v, w)w', \quad (6)$$

where

$$G_u = -\sqrt{u^3 + w}, \quad G_v = \sqrt{v^3 + w}, \quad G_w = \int_u^v \frac{1}{2\sqrt{t^3 + w}} \, dt. \quad (7)$$

Put $u = x^4$, $v = x^3$ and $w = x^2$ into the equations (6) and (7) to get

$$F(x) = G(x^4, x^3, x^2),$$

and

$$F'(x) = -4x^3 \sqrt{x^12 + x^2} + 3x^2 \sqrt{x^9 + x^2} + \int_{x^4}^{x^3} \frac{x}{\sqrt{t^3 + x^2}} \, dt.$$

From this we immediately find

$$F'(0) = 0 \text{ and } F'(1) = -\sqrt{2}.$$