Due Date: July 5, 2010, Monday NAME:...

Time: 10:30

Ali Sinan Sertöz STUDENT NO:...

Math 102 Calculus II – Homework I – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

PLEASE READ:

Check that there are 4 questions on your booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Q-1) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^5+y^6}{(x^2+y^2)^\alpha} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find all value of $\alpha \in \mathbb{R}$ such that both $f_x(0,0)$ and $f_y(0,0)$ exist. Calculate $f_x(0,0)$ and $f_y(0,0)$ for all such values of α.

Solution:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} x^{4-2\alpha} = \begin{cases} 1 & \text{if } \alpha = 2, \\ 0 & \text{if } \alpha < 2, \\ \text{No Limit} & \text{if } \alpha > 2. \end{cases}$$

Similarly,

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} y^{5-2\alpha} = \begin{cases} 1 & \text{if } \alpha = 5/2, \\ 0 & \text{if } \alpha < 5/2, \\ \text{No Limit} & \text{if } \alpha > 5/2. \end{cases}$$

Both limits exist if and only if $\alpha \leq 2$.

If $\alpha = 2$, then $f_x(0,0) = 1$, $f_y(0,0) = 0$.

If $\alpha < 2$, then $f_x(0,0) = 0$, $f_y(0,0) = 0$.

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Q-2) Assume that $3z + x + y^2 + xz^3 = 13$ defines z as a C^2 function of x and y around the point $(x, y, z) = (3, 2, 1)$. Find the values of z_x, z_y, z_{xy}, z_{yx}, z_{xx} and z_{yy} at the point $(x, y, z) = (3, 2, 1)$.

Solution:

Differentiate both sides of $3z + x + y^2 + xz^3 = 13$ implicitly with respect to x, taking y as the other independent variable and z as a differentiable function of x and y, to get

$$3z_x + 1 + z^3 + 3xz^2 z_x = 0, \tag{1}$$

which gives

$$z_x = -\frac{1 + z^3}{3(1 + xz^2)}. \tag{2}$$

Putting $(x, y, z) = (3, 2, 1)$ into the equation (1) or (2), we get

$$z_x = -\frac{1}{6}.$$

Similarly we get

$$z_y = -\frac{2y}{3(1 + xz^2)}, \tag{3}$$

and

$$z_y = -\frac{1}{3}.$$

Now using any of the equations (1),(2) or (3), differentiating implicitly and putting $(x, y, z, z_x, z_y) = (3, 2, 1, \ -1/6, \ -1/3)$, we get

$$z_{xx} = \frac{1}{24}, \quad z_{yy} = -\frac{1}{3}, \quad z_{xy} = z_{yx} = 0.$$

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Q-3) Let S be the surface in $\mathbb{R}^3$ given by $f(x, y, z) = 0$ where $f(x, y, z) = 1+x^2+y^4-z$. Let $p_0 = (1/2, y_0, z_0)$ be a point on the surface such that the tangent plane to the surface S at p_0 passes through the origin. Find z_0.

Solution:

$$\nabla f(x, y, z) = (2x, 4y^3, -1).$$

$$\nabla f(p_0) = (1, 4y_0^3, -1).$$

Equation of the tangent plane to S at p_0 is

$$(1, 4y_0^3, -1) \cdot (x - \frac{1}{2}, y - y_0, z - z_0) = 0.$$

This passes through the origin so $(x, y, z) = (0, 0, 0)$ satisfies this equation giving

$$-\frac{1}{2} - 4y_0^4 + z_0 = 0. \qquad (4)$$

Since $p_o = (1/2, y_0, z) =_0)$ is on the surface, we also have

$$\frac{5}{4} + y_0^4 - z_0 = 0. \qquad (5)$$

Adding equations (4) and (5) we get

$$y_0^4 = \frac{1}{4}.$$

Putting this value into equation (4) or (5) we get

$$z_0 = \frac{3}{2}.$$

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Q-4) Let $F(x) = \int_{x^4}^{x^3} \sqrt{t^3 + x^2}\, dt$. Calculate $F'(x)$ and find explicitly the values of $F'(0)$ and $F'(1)$.

Hint: Assume that you can differentiate under the integral sign; see the last few problems at the end of the section on "The Chain Rule" of Thomas' Calculus.

Solution:

Let $G(u, v, w) = \int_u^v \sqrt{t^3 + w}\, dt$ where u, v and w are functions of x.

Then

$$\frac{d}{dx} G(u, v, w) = G_u(u, v, w)u' + G_v(u, v, w)v' + G_w(u, v, w)w', \tag{6}$$

where

$$G_u = -\sqrt{u^3 + w}, \quad G_v = \sqrt{v^3 + w}, \quad G_w = \int_u^v \frac{1}{2\sqrt{t^3 + w}}\, dt. \tag{7}$$

Put $u = x^4$, $v = x^3$ and $w = x^2$ into the equations (6) and (7) to get

$$F(x) = G(x^4, x^3, x^2),$$

and

$$F'(x) = -4x^3\sqrt{x^{12} + x^2} + 3x^2\sqrt{x^9 + x^2} + \int_{x^4}^{x^3} \frac{x}{\sqrt{t^3 + x^2}}\, dt.$$

From this we immediately find

$$F'(0) = 0 \text{ and } F'(1) = -\sqrt{2}.$$