Q-1) Calculate $\lim_{n\to\infty} a_n$, where $a_n = \dfrac{(n!)^2}{(n^n)(6\cdot 13\cdot 20\cdots(7n-1))}$.

Solution: Observe that $\dfrac{k}{7k-1} < \dfrac{1}{6}$ for $k \geq 1$, and $\dfrac{k}{n} \leq 1$ for $1 \leq k \leq n$. Then

$$a_n = \left(\frac{1\cdot 2\cdots n}{n\cdot n\cdots n}\right)\left(\frac{1}{6}\cdot\frac{2}{7}\cdots\frac{n}{7n-1}\right) < (1)\left(\frac{1}{6}\right)^n.$$

This shows that $\lim_{n\to\infty} a_n = 0$.

Another solution is as follows.

Consider the series $\sum_{n=1}^{\infty} a_n$. Use the ratio test:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n+1}{7n+6}\frac{1}{(1+\frac{1}{n})^n} = \frac{1}{7e} < 1,$$

so the series converges. This means in particular that the general term goes to zero. Hence $\lim_{n\to\infty} a_n = 0$